I. The Syzygetic Pencil of Cubics with a new Geometrical Development of its Hesse Group, G_{216}.

II. The Complete Pappus Hexagon.

DISSERTATION

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

CHARLES CLAYTON GROVE

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INTRODUCTION.

1. The course of lectures by Prof. Frank Morley during the winter of 1903–4 on cubic curves suggested this dissertation and prepared me to carry on the research. The trend was largely determined by an incidental question by Prof. A. Cohen as to the groups involved in the system of conics which I had just presented to the Mathematical Seminary. The interest and valuable sug. gestions of Dr. A. B. Coble in the carrying on of the work are gratefully acknowledged.

2. The close connection between the Hesse group and the syzygetic pencil of cubics makes it necessary to say at least something about this pencil of curves. Without attempting even an outline of the theory, I present in Section I. only such matter as is needed later, besides some new facts concerning the pencil and a figure showing the appearance of some noteworthy and specially related cubics of the pencil. No figure seems ever to have been published except that in connection with the paper of Prof. Morley in the Proceedings of the London Math-Society, Ser. 2, Vol. 2, Part 2, which shows arbitrarily selected cubics. The initial and all but the closing work leading to that figure was done by me. Therefore, I present a figure of the pencil herein, also one of the corresponding polar-reciprocal range of line cubics.

Section II. shows how to derive a closed system of thirty-six conics analogous to the conic of Section I. as to which the pencil and range are polar-reciprocal. It also discusses the action of the polarities of these conics upon the four inflexional triangles, and presents some history of similar considerations.

In Section III. there is given a brief history of the attempts to determine all finite groups of transformations, and in particular an account of the Hesse Group of 216 collineations. Further, we derive and write down the matrices of these collineations by means of the closed system of thirty-six conics, which are here differently defined than in Section II. and are given accordingly. All the subgroups are found and discussed. The collineations are finally classified as to periodicity.

Section IV. treats of triangles in other perspective forms than six-fold as are the inflectional triangles above. As a second way to secure triangles in threefold perspective, also some in two- and one-fold perspective, we develop what we call the Complete Pappus Hexagon, in its dualistic forms and deduce a number of theorems connected with it.

I.

THE SYZYGETIC PENCIL. DRAWINGS OF THE PENCIL AND RANGE.

1. The name SYZYGETIC[1] is given to the pencil of cubics determined by a cubic f and its Hessian Δ,

$$\varkappa f + \lambda \Delta = 0, \tag{1}$$

which has the same nine inflexions for all its cubics, the intersections of the two cubics f and Δ, and so has the same four inflexional triangles.

It is well known that any non-singular cubic may be brought in four ways, as shown for example by Weber,[2] into Hesse's canonical form

$$x_1^3 + x_2^3 + x_3^3 + 6\,m x_1 x_2 x_3 = 0. \tag{2}$$

This simply means that the cubic has been referred to one of its inflexional triangles as reference triangle. The Hessian covariant of the form (2) is

$$\Delta \equiv -\,m^2\,(x_1^3 + x_2^3 + x_3^3) + (1 + 2\,m^3)\,x_1 x_2 x_3\,. \tag{3}$$

Its vanishing gives the Hesse Cubic or Hessian. Thus if m is the parameter of the cubic (2) and m' is that of its Hessian, we have $6\,m' = -\dfrac{1 + 2\,m^3}{m^2}$; and form (2) for all values of m from $-\infty$ to $+\infty$ gives the pencil as well as form (1).

The relation between m and m' shows that *each cubic of the pencil has but one Hessian but is Hessian to three cubics of the pencil.*

Choosing one of the inflexional triangles as reference triangle, we readily calculate in turn,

The coördinates of the nine inflexions,

Their arrangement on the sides of each of four triangles,[3]

The equations of the sides and opposite vertices of these triangles,

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[1] Clebsch-Lindemann: Leçons sur la Géométrie, II, p. 230.

[2] Lehrbuch der Algebra, 2. Aufl., II, §§ 106, 107; ss. 399–404.

[3] C.-L.: II, (7) p. 233.

The polar conics of the inflexions, which are in each case[1] two right lines, viz., the inflexional tangent and the harmonic polar of the inflexion.

For future reference we give the sides of the inflexional triangles:

A.	B.	C.	D.	
$x_1 = 0.$	$x_1 + x_2 + x_3 = 0.$	$\omega^2 x_1 + x_2 + x_3 = 0.$	$\omega x_1 + x_2 + x_3 = 0.$	
$x_2 = 0.$	$x_1 + \omega^2 x_2 + \omega x_3 = 0.$	$x_1 + \omega^2 x_2 + x_3 = 0.$	$x_1 + \omega x_2 + x_3 = 0.$	(4)
$x_3 = 0.$	$x_1 + \omega x_2 + \omega^2 x_3 = 0.$	$x_1 + x_2 + \omega^2 x_3 = 0.$	$x_1 + x_2 + \omega x_3 = 0.$	

The equations of the vertices respectively *opposite* are given by exchanging ξ for x and interchanging ω and ω^2. ω and ω^2 are the complex cube roots of unity.

2. Some Particular Cubics of the Pencil.

(*a*) The cubics whose parameter m is respectively ∞, $-\frac{1}{2}$, $-\frac{1}{2}\omega^2$, $-\frac{1}{2}\omega$ are the four inflexional triangles in the order as obtained above, of which[2] two are real and two imaginary. The polar line[3] as to these triangles of any point y has coordinates respectively,

	x_1	x_2	x_3
A:	$y_2 y_3$	$y_3 y_1$	$y_1 y_2.$
B:	$y_1^2 - y_2 y_3$	$y_2^2 - y_3 y_1$	$y_3^2 - y_1 y_2.$
C:	$y_1^2 - \omega^2 y_2 y_3$	$y_2^2 - \omega^2 y_3 y_1$	$y_3^2 - \omega^2 y_1 y_2.$
D:	$y_1^2 - \omega y_2 y_3$	$y_2^2 - \omega y_3 y_1$	$y_3^2 - \omega y_1 y_2.$

The determinant formed from any three rows of these coefficients vanishes identically, therefore the four polars pass through a common point, so we say,

The four polar lines of any point as to the four inflexional triangles meet in a point, or also

Any two of the inflexional triangles are apolar as seen from any point of the plane, and are thus syzygetic.

(*b*) The first of two covariant cubics of the pencil is the Hessian, equation (3), p. 5. The second is the Cayleyan of the cubic (2). The polar conic as to this cubic of a point y is

$$y_1(x_1^2 + 2mx_2x_3) + y_2(x_2^2 + 2mx_3x_1) + y_3(x_3^2 + 2mx_1x_2) = 0.$$

Considering the y's as parameters, this is a *net of polar conics.* By inspection, we see that the conic $m\xi_1^2 - \xi_2\xi_3$ is apolar with the three conics of the net;

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[1] C.-L.: II, p. 227; Salmon: Higher Plane Curves, 3. Ed., §§ 74, 170, pp. 59, 146.

[2] C.-L.: II, p. 230, also pp. 239, 310.

[3] H. P. C., § 165, p. 143.

likewise the conics $m\xi_2^2 - \xi_3\xi_1$, and $m\xi_3^2 - \xi_1\xi_2$. So the net has a corresponding *web of apolar conics*, given by the equation

$$\eta_1(m\xi_1^2 - \xi_2\xi_3) + \eta_2(m\xi_2^2 - \xi_3\xi_1) + \eta_3(m\xi_3^2 - \xi_1\xi_2) = 0,$$

where the η's are parameters.

The contravariant or Jacobian of the web is the Cayleyan:

$$\begin{vmatrix} 2m\xi_1 & -\xi_3 & -\xi_2 \\ -\xi_3 & 2m\xi_2 & -\xi_1 \\ -\xi_2 & -\xi_1 & 2m\xi_3 \end{vmatrix} = 0,$$

or

$$m(\xi_1^3 + \xi_2^3 + \xi_3^3) + (1 - 4m^3)\xi_1\xi_2\xi_3 = 0,$$

which is a cubic of the range[1] enveloped by the ∞^2 lines composing the degenerate conics, the first polars of points along the Hessian of the cubic.

Here as in the case of the Hessian, we see that *each cubic of the range has but one Cayleyan but it is Cayleyan to three cubics.*

(c) The simplest invariant is found by operating with the Cayleyan on the cubic.[2] It is the QUARTIC INVARIANT denoted in Salmon by S. By operating thus and dividing[3] by 24 we have

$$S \equiv m(1 - m^3).$$

The vanishing of this invariant gives the parameters of four curves of the pencil called the *equianharmonic cubics*:

$$\begin{aligned} &S_1: x_1^3 + x_2^3 + x_3^3 = 0. && S_3: x_1^3 + x_2^3 + x_3^3 + 6\omega^2 x_1x_2x_3 = 0. \\ &S_2: x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3 = 0. && S_4: x_1^3 + x_2^3 + x_3^3 + 6\omega x_1x_2x_3 = 0. \end{aligned} \quad (5)$$

This name is given because the constant anharmonic ratio of the four tangents[4] drawn from a point of the curve tangent to the curve itself is equianharmonic in these four cases.

(d) By operating with the Cayleyan on the Hessian we have the SEXTIC INVARIANT, T of Salmon.[5] It is

$$T \equiv 1 - 20m^3 - 8m^6.$$

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[1] C.-L.: II, p. 244.

[2] By operating with a line-form, as of the Cayleyan, upon a point-form, as of the cubic (2) called f, we mean that the ξ's are taken as partial differential operators. The equation of the Cayleyan means in this process

$$m\left(\frac{\partial^3 f}{\partial x_1^3} + \frac{\partial^3 f}{\partial x_2^3} + \frac{\partial^3 f}{\partial x_3^3}\right) + (1 - 4m^3)\frac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_3}.$$

[3] Salmon: H. P. C., § 220, p. 191.

[4] C.—L.: II, pp. 325, 326.

[5] H. P. C., § 221, p. 193, also Weber: II, s. 406 (7).

The curves whose parameters are given for $T = 0$, are the six *harmonic* cubics of the pencil, called so since the ratio of the four tangents to the curve from points on it is harmonic.[1]

With this amount of introduction and number of references necessary to prepare the student to read the whole most profitably, and with the equations at hand to which reference must later be made, we pass now to

The Drawing of the Syzygetic Pencil.

3. As a far more convenient form of the cubics of the pencil for purposes of construction, we transform the equation of the pencil so as to have inflexions at the circular imaginary points I and J, by putting

$$x_1 = x + y + 1, \qquad x_2 = -(x - 1), \qquad x_3 = -(y - 1), \tag{6}$$

where x and y are conjugate coördinates.

By this transformation, equation (2) of the pencil becomes

$$xy(x + y) + \frac{2(1 - m)}{1 + 2m}(x^2 + xy + y^2) + 1 = 0,$$

or it is of the form

$$xy(x + y) + \mu(x^2 + xy + y^2) + 1 = 0, \tag{7}$$

where
$$\mu = \frac{2(1 - m)}{1 + 2m}, \text{ and } m = \frac{2 - \mu}{2(1 + \mu)}.$$

4. The Special Cubics of § 2.

(*a*) The four inflexional triangles (4) become in conjugate coördinates,

$$\begin{aligned} \text{A}: &\quad (x + y + 1)(x - 1)(y - 1) = 0. \\ \text{B}: &\quad x^2 + xy + y^2 - (x - \omega y)(x - \omega^2 y) = 0. \\ \text{C}: &\quad xy(x + y) - \omega(x^2 + xy + y^2) + 1 = 0. \\ \text{D}: &\quad xy(x + y) - \omega^2(x^2 + xy + y^2) + 1 = 0. \end{aligned} \tag{8}$$

Our former reference triangle A is seen thus to be now the line through the points ω and ω^2 on the unit-circle, and the point $x = 1$ taken twice.

Since the cube terms are lacking in B, the cubic consists of the line at infinity, also, as readily seen, of the lines through the origin and ω and ω^2 respectively. The representable parts are shown in the figure of the pencil. The three dash lines are the real harmonic polars, the full lines are the degenerate cubics.

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[1] C.-L.: II, p. 326.

(*b*) The Hessian by direct calculation on (7) is

$$\frac{1}{6}\begin{vmatrix} 2y+2\mu & 2(x+y)+\mu & \mu(2x+y) \\ 2(x+y)+\mu & 2x+2\mu & \mu(2y+x) \\ \mu(2x+y) & \mu(2y+x) & 6 \end{vmatrix}$$

or $$H \equiv 3\mu^2 xy(x+y) - (\mu^3+4)(x^2+xy+y^2) + 3\mu^2 = 0, \qquad (9)$$

which is of the form of the cubic (7) with its parameter

$$\mu' \text{ given by } \mu' = -\frac{\mu^3+4}{3\mu^2}.$$

Therefore equation (7) likewise gives the syzygetic pencil for all values of μ from $-\infty$ to $+\infty$ as well as equation (1).

The Cayleyan is calculated as follows:

The polar conic as to the cubic (7) of a point p is

$$p_1[2xy+y^2+\mu(2x+y)] + p_2[x^2+2xy+\mu(x+2y)] + p_3[\mu(x^2+xy+y^2)+3] = 0.$$

Regarding the p's as parameters this is a *net of polar conics.*

To find the *web of apolar conics* we take the general line-conic

$$a\xi^2 + b\eta^2 + c\zeta^2 + d\xi\eta + e\eta\zeta + f\zeta\xi = 0,$$

and operate separately on the point conics of the net. Thus we obtain

$$2b + 2d + 2\mu f + \mu e = 0,$$
$$2a + 2d + \mu f + 2\mu e = 0,$$
$$2\mu a + 2\mu b + \mu d + 6c = 0.$$

We desire apolar conics analogous in form respectively to those of the net and so we first take $a = 0$, $b = d = 1$, whence, by substituting in the three equations above and in the general line-conic, we get the conic

$$2\mu\xi\eta + 2\mu\eta - 4\xi - \mu^2 = 0.$$

Second, put $b = 0$, $a = d = 1$, and we get the conic

$$2\mu\xi^2 + 2\mu\xi\eta - 4\eta - \mu^2 = 0.$$

Third, put $a = 1$, $e = f = 0$, and the corresponding conic is

$$2(\xi^2 - \xi\eta + \eta^2) - \mu = 0.$$

By these three we may write the apolar web of line-conics,

$$\Pi_1[2\mu\xi\eta + 2\mu\eta^2 - 4\xi - \mu^2] + \Pi_2[2\mu\xi^2 + 2\mu\xi\eta - 4\eta - \mu^2] + \Pi_3[2(\xi^2 - \xi\eta + \eta^2) - \mu] = 0.$$

The Jacobian of this web is by direct calculation

$$\begin{vmatrix} \mu\eta - 2 & \mu(\xi + 2\eta) & -(2\xi + \mu^2) \\ \mu(2\xi + \eta) & \mu\xi - 2 & -(2\eta + \mu^2) \\ 2\xi - \eta & -\xi + 2\eta & -\mu \end{vmatrix},$$

or $$2\mu(\xi^3+\eta^3)-3\mu\xi\eta(\xi+\eta)+(\mu^3-2)(\xi^2-\xi\eta+\eta^2)-\mu^2(\xi-\eta)-\mu=0, \quad \textbf{(10)}$$

which is the Cayleyan of the cubic (7).

(*c*) By operating with the Cayleyan (10) on the cubic (7) we have the *quartic invariant*

$$S = \mu^4 - 8\mu,$$

whose vanishing gives the parameters of the four equianharmonic cubics of the pencil, viz., $\mu = 0,\ 2,\ 2\omega,\ 2\omega^2$. That two of these are real is seen from the parameters.

(*d*) By operating with the Cayleyan on the Hessian (9) we obtain the *sextic invariant*

$$T = 8 - 20\mu^3 - \mu^6,$$

whose vanishing gives the parameters of the six harmonic cubics, viz., the roots of $\mu^3 = -10 \pm 6\sqrt{3}$. For the positive and the negative sign, there are in each case one real and two imaginary roots, or there are two systems of three values each of the harmonic ratio. The two real roots are

$$-(1-\sqrt{3}) \text{ and } -(1+\sqrt{3}).$$

5. The nine inflexions and harmonic polars mentioned in § 1, pp. 5, 6 are in conjugate coördinates as follows, with the positions on the figure as indicated respectively —

Inflexions.	*Point on figure.*	*Its Harmonic Polar.*	*The Line on the Figure.*	
1.	At ∞ on $\overline{\omega\,\omega^2}$	$x - y = 0$	Axis of reals	**(11)**
2.	I	$x + 2y = 0$	Circular imaginary rays from the origin	
3.	J	$2x + y = 0$		
4.	ω^2	$x - \omega^2 y + \omega^2 - 1 = 0$	Line $\overline{1, -\omega^2}$	
5.		$x - \omega^2 y - \omega + 1 = 0$	A cir. imag. ray through ω	
6.		$x - \omega^2 y - \omega^2 + \omega = 0$	" " " " " ω^2	
7.	ω	$x - \omega y + \omega - 1 = 0$	Line $\overline{1, -\omega}$	
8.		$x - \omega y - \omega^2 + 1 = 0$	A cir. imag. ray through ω^2	
9.		$x - \omega y - \omega + \omega^2 = 0$	" " " " " ω	

By projecting the pencil into this form, we have the line at infinity as one real side of an inflexional triangle, and one real inflexion at infinity. That leaves[1] to appear on the figure three sides of inflexional triangles, two inflexions and three harmonic polars.

Towards a better understanding of the syzygetic pencil and the drawing of it we analyze it further and present first amongst our findings.

6. THE ASYMPTOTES.

Take the line $x + y = \lambda$, which is perpendicular to the axis of reals at a distance $\frac{1}{2}\lambda$ from the origin; that is, the line in which the reflexion of the origin is the point λ. It cuts the cubic (7) where

$$\lambda x(\lambda - x) + \mu(x^2 - \lambda x + \lambda^2) + 1 = 0.$$

Since the cube terms vanish, the line meets the cubic at infinity. If next we put $\lambda = \mu$, we have only

$$\mu^3 + 1 = 0.$$

Hence, since the square terms also vanish, the line $x + y = \mu$ is tangent to the cubic at infinity, in direction perpendicular to the axis of reals and is the asymptote.

The point at infinity on the line $\overline{\omega\omega^2}$, perpendicular to the axis of reals is as above noted one of the real inflexions, hence *the asymptote just found is an inflexional tangent as well.*

Thus when μ is given, the asymptote of the cubic of the pencil for that particular μ as parameter is also given as the perpendicular to the axis of reals at the distance $\frac{1}{2}\mu$ from the origin.

7. By taking the first polar of the inflexion ω as to the general cubic (7) we find it breaks up into the two linear factors

$$\{x - \omega y + \omega - 1\}\ \{\omega^2(x - \omega y - \omega + 1) + \mu(x - \omega^2 y)\} = 0,$$

the harmonic polar and flex-tangent respectively.

The latter cuts the asymptote of the cubic where

$$-(1 + \mu)(\omega x + \omega^2\mu - \omega^2 + 1) = 0.$$

The harmonic polar of ω^2 [see equations (11), p. 10] cuts the asymptote where $-(\omega x + \omega^2\mu - \omega^2 + 1) = 0$. Therefore, *we may draw the flex-tangents to any cubic of pencil by drawing from either inflexion,* ω or ω^2, *to the point where the harmonic polar of the other,* ω^2 or ω, *cuts the asymptote of that cubic.*

[1] C.-L.: II, p. 235 on the real parts of the figure.

8. Intersections of the Cubics with the Axis of Reals.

The general cubic (7) cuts the axis of reals where

$$2x^3 + 3\mu x^2 + 1 = 0.$$

The discriminant (Weber: I, s. 273) of this equation is

$$D \equiv -108(\mu^3 + 1),$$

which shows that if

1. $\mu < -1$, then $D > 0$, and the equation has three real distinct roots.
2. $\mu = -1$, then $D = 0$, " " " " one real repeated root.
3. $\mu > -1$, then $D < 0$, " " " " one real, two imaginary roots.

As to the cubics this says that

1. for $\mu < -1$, the cubic cuts the axis of reals in three points and hence is, in general, of the *bipartite* type.

2. for $\mu = -1$, there is one actual intersection and an acnode on the axis, as was noted in § 4 (*a*), p. 8.

3. for $\mu > -1$, there is but one real intersection and the cubic is of the *unipartite* type.

The study of these intersections thus enables us to classify the cubics from the parameter.

9. Besides the aids in constructing the syzygetic pencil furnished by these facts as to the asymptotes, the inflexional tangents, and the intersections of the cubics with the axis of reals, we present, as the final means of facilitating the construction, the drawing of a number of circles concentric with the unit-circle and the calculation of the intersections of the cubics with these circles.

The cubic (7) cuts the circle $xy = \rho^2$ where

$$\mu\left(x + \frac{\rho^2}{x}\right)^2 + \rho^2\left(x + \frac{\rho^2}{x}\right) + 1 - \rho^2\mu = 0.$$

Since x and y are conjugate complex coördinates, $x + \frac{\rho^2}{x} = c$ is a right line perpendicular to the axis of reals at a distance $\frac{1}{2}c$ from the origin. Therefore, the cubic cuts the circle where this perpendicular does, for values of c which are the roots of the equation in $\left(x + \frac{\rho^2}{x}\right)$.

The location of the perpendicular is given by

$$x = \frac{-\rho^2 \pm \sqrt{4\mu(\rho^2\mu - 1) + \rho^4}}{4\mu}. \qquad (12)$$

10. Two interesting matters of analysis will be noted before presenting the drawing. The Hessian cubic whose parameter is μ' is Hessian to three cubics whose parameters are the roots of

$$\mu^3 + 3\mu'\mu^2 + 4 = 0. \qquad \textbf{(13)}$$

Their asymptotes cut the axis of reals at $\frac{1}{2}\mu$ resp. The Hessian itself cuts the axis of reals at points given by the roots of $2x^3 + 3\mu' x^2 + 1 = 0$.

If we put in this equation $x = \frac{1}{2}\mu$, we obtain identically the former equation. Therefore, the Hessian cuts the axis where the asymptotes of its curves do, and these asymptotes are its tangents on the axis of reals. Or, for this form of the pencil, we may say, *the Hessian is tangent on the axis of reals to the asymptotes of its curves.*

Since the axis of reals is the harmonic polar of the real inflexion at infinity and the asymptotes are flex-tangents at this inflexion, we may state the theorem projectively:

The three flex-tangents of the three cubics with a common Hessian, at any one inflexion, touch this Hessian in three points on the harmonic polar of the inflexion considered.

This is an extension of theorems by Alfred Clebsch[1] and Peter Muth.[2]

The discriminant of equation (13) is $D = -432(\mu'^3 + 1)$. Thus, if the *Hessian* is *bipartite*, i. e., if $\mu' < -1$, there are *three real unipartite cubics* of which it is Hessian; for the roots of equation (13) are then all real, two positive and one negative but greater than -1.

If $\mu' = -1$, its cubic is Hessian of the cubics whose parameters are $-1, 2, 2$. That is, it is Hessian of itself and of one of the two real harmonic cubics twice over.

If the *Hessian is unipartite* it is the *Hessian of one real, bipartite cubic.*

Since our names unipartite and bipartite, following Salmon, are actually names of the cubics von Staudt calls resp. odd- and even circuit cubics, and since[3] by no projection does a non-singular cubic change its class of odd- or even-circuit, these facts as to the Hessian and its cubics remain for all real projections.

11. The two real harmonic cubics are readily seen to be mutually Hessian and cubic, for the parameters $-(1 + \sqrt{3})$ and $-(1 - \sqrt{3})$ may be interchangeably μ and μ' and satisfy equation (13).

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[1] Ueber die Wendetangenten der Curven dritter Ordnung; Crelle Journal 58, s. 232.

[2] Ueber ternäre Formen, u. s. w.; Inaug. Diss.—Giessen, 1890, s. 15.

[3] C.-L.: II, end of foot-note p. 223.

Therefore, the flex-tangents of each touches the other and so, in our form, each cuts the axis of reals at the asymptote of the other. Compare Clebsch in Crelle Journal, Bd. 58, ss. 238, 239.

12. For the form (7) of the pencil, the two invariants are got very nicely as the invariants of the quartic[1] giving the intersections of the four tangents from the real inflexion at infinity.

As shown in § 6, p. 11, the flex-tangent at infinity cuts the axis at $x = \frac{1}{2}\mu$, and in § 8, p. 12, the cubic cuts the axis at $2x^3 + 3\mu x^2 + 1 = 0$. Therefore, the intersections of the four tangents from the real inflexion at infinity are given by the quartic $(2x^3 + 3\mu x^2 + 1)(2x - \mu) = 0$, or

$$4x^4 + 4\mu x^3 - 3\mu^2 x^2 + 2x - \mu = 0.$$

The invariants[2] of this quartic are those of the cubic, given in § 4, (*c*) and (*d*), to within a numerical factor.

The Syzygetic Range of Cubics.[3]

13. The invariant parts of the syzygetic pencil just studied are well known and are known to correspond dualistically throughout. In view of this correspondence it is observed that the polarity arising from the conic

$$x_1^2 + x_2^2 + x_3^2 = 0, \qquad (14)$$

namely, $x_1 = \xi_1, \quad x_2 = \xi_2, \quad x_3 = \xi_3,$

sends each part into its corresponding part, also the point-cubics of the pencil into the line-cubics of range,

$$\xi_1^3 + \xi_2^3 + \xi_3^3 + 6m\xi_1\xi_2\xi_3 = 0. \qquad (15)$$

The conic (14) transformed into conjugate coördinates becomes

$$2(x^2 + xy + y^2) + 3 = 0,$$

which equation shows the conic to be an hyperbola with the inflexional lines (see *B*, (8), p. 8) $x - \omega y = 0$ and $x - \omega^2 y = 0$ as asymptotes, with vertices at $\pm \frac{1}{2} i \sqrt{6}$. Transformed to rectangular coördinates with the axis of reals as the X-axis, it becomes $6X^2 - 2Y^2 + 3 = 0$; and with the asymptotes as axes it is $XY = -\frac{3}{2}$. From this equation we easily construct the conic by considering the equality of segments of chords contained between the curve and its asymptotes.

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[1] Salmon: H. P. C., § 228, p. 199.

[2] Weber: I, § 70, s. 230; II, § 108, s. 406.

[3] C.-L.: II, p. 244 sq.

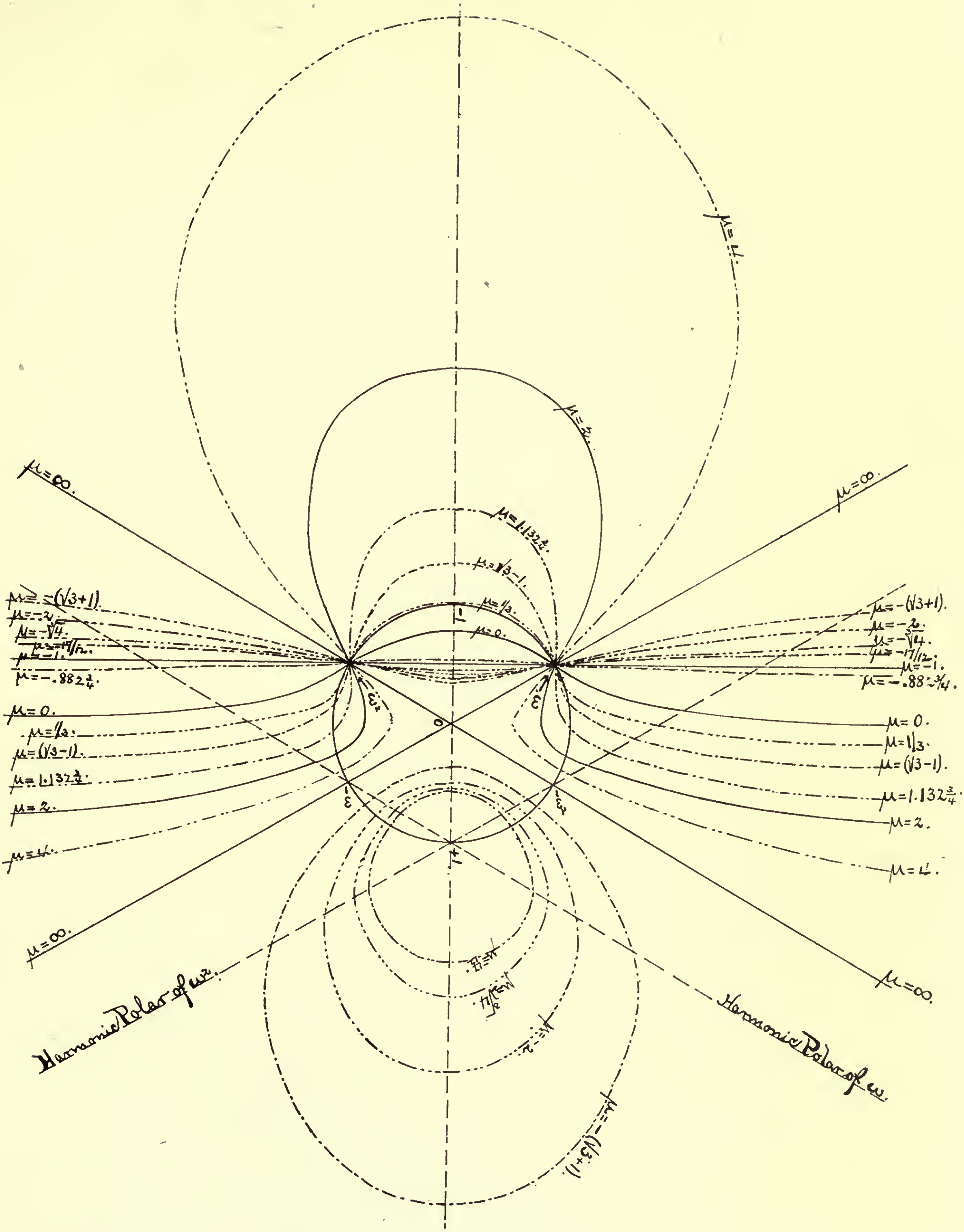

Fig. 1.—The Syzygetic Pencil of Cubics.

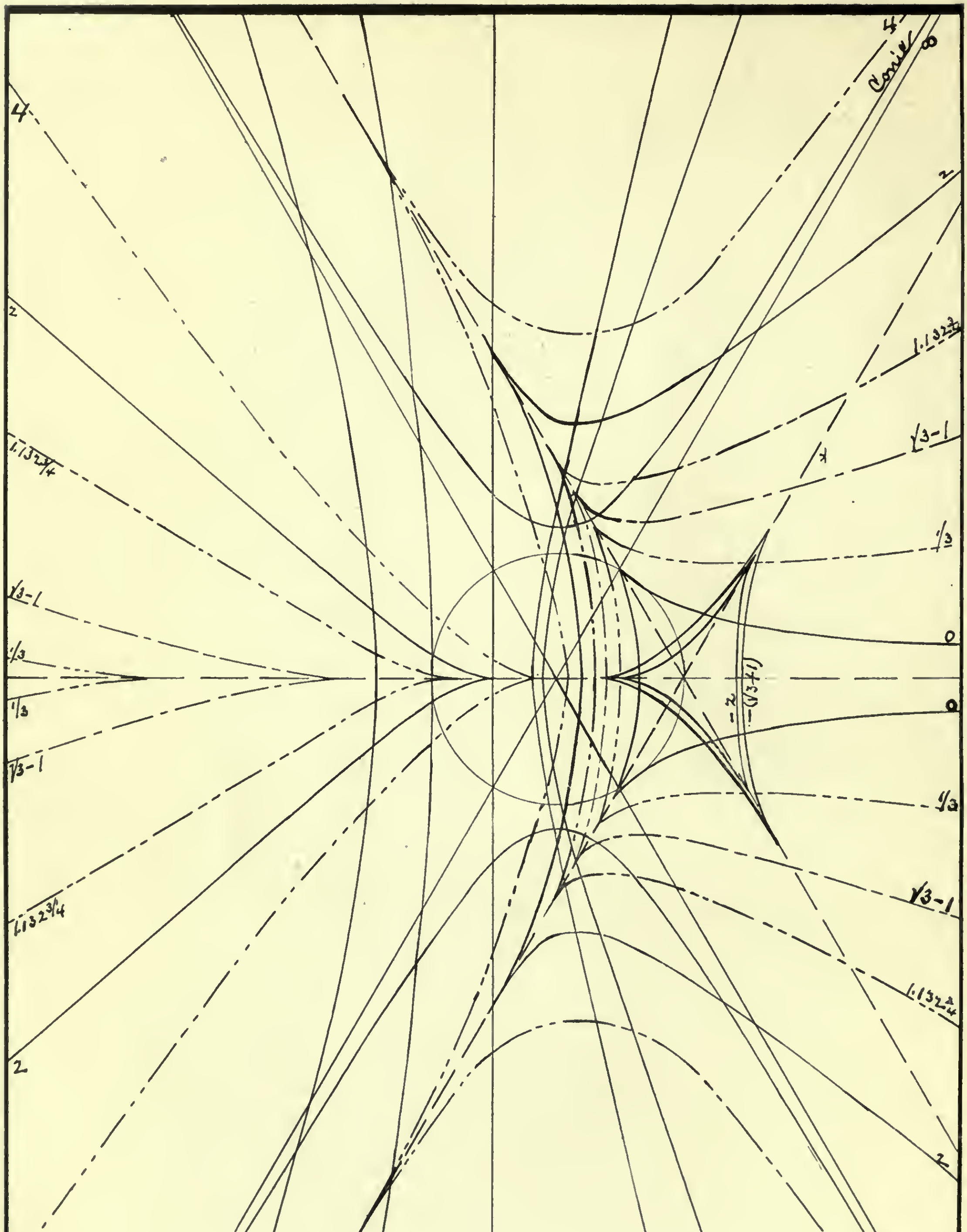

Fig. 2.—The Syzygetic Range, polar reciprocal of Pencil as to Conic

By the polar-reciprocal process as to this conic we deduce the line-cubics of the syzygetic range from the point-cubics of the pencil. The harmonic polars are seen to become the cusp-tangents common to all the curves of the range.

Explanation of the Figures.

The Syzygetic Pencil. The inflexional lines are solid. The harmonic polars are dash lines. All cubics are marked with their parameters. Specially related ones are the same kind of lines.

The cubics represented are:

1. Two equianharmonic, $\mu = 0$ and 2.

2. Two harmonic, $\mu = -(1 - \sqrt{3})$ and $-(1 + \sqrt{3})$. They cut the axis of reals respectively at $-\frac{1}{2}(1 + \sqrt{3})$ and $-\frac{1}{2}(1 - \sqrt{3})$; the former is unipartite, the latter bipartite. Their inflexional tangents from ω and ω^2 are drawn in fine dotted lines.

3. The cubic for which $\mu = -\frac{17}{12}$ is Hessian of three for which μ is resp. 4, $1.132\frac{3}{4}$, $-0.882\frac{1}{4}$. These four are drawn alike.

4. The cubic $\mu = \frac{1}{3}$ is Hessian of but one real cubic for which $\mu = -2$.

5. The equianharmonic cubic, $\mu = 0$, is also Hessian of but one real cubic for which $\mu = -\sqrt[3]{4}$. It is draw different from the others.

The Syzygetic Range. The majority of the above cubics were reciprocated in the conic shown on the figure and the polar-reciprocal line-cubics appear drawn in the same kind of lines as the corresponding point-cubics from which they were derived and are named with the same value of μ respectively.

Clebsch has a drawing (C.-L.: II, p. 243) of the range but does not profess it to have been constructed. Its form is very considerably different from the accompanying drawing.

II.

ON A CLOSED SYSTEM OF CONICS.

14. The conic (14) of § 13, p. 14, is referred to triangle A of equations (4), p. 6, as reference triangle with the first line of B (ibid.) as auxiliary line. Its effect on the parts of the syzygetic pencil as given in § 13, raised the question as to the effect of the analogous conics as to all the possible reference frames in the four triangles (4).

The equations of these analogous conics may be got by considering separately each of the four triangles as reference triangle with the nine remaining lines in turn as the auxiliary line. Thus, there are thirty-six such conics, nine for each triangle. As an example, say we wish to consider triangle B and the first side of triangle C as reference frame. Then,

$$x_1' \equiv \omega x_1 + \omega x_2 + \omega x_3 = 0,$$
$$x_2' \equiv x_1 + \omega^2 x_2 + \omega x_3 = 0,$$
$$x_3' \equiv x_1 + \omega x_2 + \omega^2 x_3 = 0,$$

whence $x_1' + x_2' + x_3' \equiv (\omega^2 + 2\omega)(\omega^2 x_1 + x_2 + x_3) = 0$ is auxiliary line as we desired. The conic $x_1'^2 + x_2'^2 + x_3'^2 = 0$ has therefore as to the original reference frame the equation

$$\omega x_1^2 + x_2^2 + x_3^2 + 2(\omega x_2 x_3 + x_3 x_1 + x_1 x_2) = 0.$$

Similarly by inspection all the others may be deduced.

The equations of these conics will be given in the next chapter where they are derived differently and their properties are defined. Their development as above was published in THE JOHNS HOPKINS UNIVERSITY CIRCULAR, January 1905, pp. 16 ff.

These thirty-six conics form a closed system for by operating two times with a polarity arising from any conic of the system we get a polarity of the system; or, the product of three polarities of the system is a polarity of the system.

It is easily shown that when a conic reciprocates a triangle into another, the two triangles are in perspective, and conversely. For two triangles in n-fold perspective there are n such conics, so for the four inflexional triangles, mutually in six-fold perspective, there should be six times ${}_4C_2$ or thirty-six conics, as there are.

15. As to their effect upon the inflexional triangles, the polarities divide into sets in two ways. First, by nines A_{1-9}, B_{1-9}, etc., they send the vertices respectively of triangles A, B, etc., into the sides opposite, and at the same time reciprocate another triangle into itself and the vertices of each of the other two triangles into the sides of the other of these two triangles. Second, they divide into sets of six each, operating as indicated in the table:

Polarities.		*Triangles.*		*Triangles.*	
A_{1-3}, B_{7-9}	send	A and B	into themselves respectively, and	C and D	into each other
A_{4-6}, C_{4-6}	"	A " C	" "	D " B	" " "
A_{7-9}, D_{1-3}	"	A " D	" "	B " C	" " "
B_{1-3}, C_{7-9}	"	B " C	" "	A " D	" " "
D_{4-6}, B_{4-6}	"	D " B	" "	A " C	" " "
C_{1-3}, D_{7-9}	"	C " D	" "	A " D	" " "

The classification shows also that besides the nine polarities which send a triangle into itself, vertices into sides opposite, there are nine others sending it into itself, but vertices into sides in anti-cyclic order as to the order of the vertices.

We shall speak of the three cyclic and the three anti-cyclic forms of perspective, meaning as indicated here,

Cyclic Forms.			*Anti-cyclic Forms.*		
1′	2′	3′	1′	2′	3′
1	2	3	1	3	2
2	3	1	2	1	3
3	1	2	3	2	1

As analysis readily shows, the projecting rays are the nine harmonic polars with four vertices of the triangles on each. The centers of perspective are the vertices of the triangles, and the axes their sides. Further, for any two of the triangles, the vertices and sides of one of the remaining two triangles are respectively the centers and axes in cyclic order for the three cyclic perspectivities, and those of the other triangle respectively in anti-cyclic order are centers and axes for the three anti-cyclic perspectivities.

16. We would call attention to two very neat papers on perspective triangles by J. Ványi.[1] He makes a slight error as to 6-fold perspective triangles by saying, "Unter den 6 Kegelschnitten giebt es höchstens vier reelle, die beiden Dreiecke sind immer imaginär," whereas our equations (4), p. 6, show one whole triangle and one side of the other may be real.

After publishing the article in the J. H. U. Circular (§ 14, p. 16), I found the following papers by S. Kantor.

In 1895 he notes[2] the 36 collineations, in connection with two triangles in 6-fold perspective. The following year he speaks[3] of 36 conics in connection with the four Hesse Triangles. He uses these conics, which are the 36 herein presented but defined differently as stated in Th. XIII and used to operate on the collineations of types 5, 6, 7 of Jordan[4] to produce the most general group of correlations which contain these types.

By finding the "*intermediate*" or Salmon's contravariant conic Φ[5] of each

[1] Archiv der Mathematik und Physik: 1882, Bd. 70, ss. 105–110; 1884, 2. R., II. T., ss. 230–234.

[2] Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene, (Berlin) s. 58.

[3] Crelle Journal, Bd. 116, ss. 176, 177.

[4] Ibid., Bd. 84, s. 92.

[5] Salmon-Fiedler: Kegs, 6. Aufl., II, s. 668.

two conics in the 36 and remembering the geometrical meaning of the intermediate, we learn the relative positions of the 36 conics. Also, this process gives us other sets of line-conics mostly of 27 each but none are the 27 having six-fold contact with a cubic of the pencil[1] as we hoped. These intermediates are not sufficiently pertinent to our subject to be given here.

III.

THE HESSE GROUP OF 216 COLLINEATIONS.

17. History and Bibliography.

The attempts to determine all finite groups of transformations run back to the work of F. Klein in the number for July 1874 of the *Sitzungsberichte der Erlanger phys.-med. Gesellschaft:* Ueber binäre Formen. Again in 1876, Math. Ann. 9:185, he sets the problem, "Alle Gruppen anzugeben, welche aus einer endlichen Anzahl von linearen Transformationen bestehen," and proceeds to determine them by the method of rotations of the regular polyhedrons into themselves respectively.

In 1876, L. Fuchs in Crelle Journal 81 : 97–142, and the following year P. Gordon in Math. Ann. 12 : 23–46, confirm the results of Klein for the binary domain by entirely different methods.

Camille Jordan in a notable memoir (1878, Crelle Journal, 84 : 89–215) tried to give *completely* the groups of both the binary and the ternary domain. He discovered the group G_{216}, mentioned ibid. p. 206 and called the Hesse group. He had noted its existence before the writing here cited as he therein states. Two years later (1880), Jordan devotes himself directly to the group problem in a memoir in the Atti della R. Accademia d. Scienze Fisiche e Math. Società Reale di Napoli, Vol. VIII, No. 11, pp. 1–41. It is worthy of note that Jordan did not find the simple group G_{168}, discovered by Klein [Math. Ann. (1879) 14 : 428–471].

Next, in 1887, the Hessian group is considered by Alexander Witting in his inaug. diss. (Göttingen, 58, S. 8) by the use of hyperelliptic functions.

Heinrich Maschke gives the fullest treatment of the subject. See Math. Ann. 29 : 157 ff.; Nachrichten d. K. Gesellschaft d. Wiss. zu Göttingen, (1888) Nr. 5, ss. 78 ff.; and especially in connection with his presentation of the group of 51840 transformations (Math. Ann. 33 : 317–344) in 1889.

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[1] Ibid.: Höhern Eb. Kur., § 156, s. 173.

Again in 1889 we have an important memoir by H. Valentiner: "De endlige Transformations-Gruppers Theori," in Copenhagen · K. Danske Videnskabs. Selskab., Naturvid. og Math. 6. Raekke V. : 2, pp. 67–204; French résumé pp. 205–235. Valentiner shows rather clearly that he did not know of the memoire of Jordan and on pp. 151, 222, denies the existence of the group of 216 collineations. He demonstrates (p. 69) the possibility of a group of 72 transformations containing ones of 2nd, 3rd and 4th order, where the transformations of the 4th order by sixes have a common second power. Later he shows that the group really exists. He discovered the group G_{360} and presents it on pp. 191–198, French pp. 231–233.

A. Wieman refers to the Hessian group in connection with his paper in Math. Ann. 47 : 531–556: "Ueber eine einfache Gruppe von 360 ebenen Collineationen," (1896), and of course in his article on "Endliche Gruppen Linearer Substitutionen" (1900) in Encyklopädie der Mathematischen Wissenschaften, Bd. I, H. 5, 3 f. (See p. 528.)

S. Kantor's paper referred to in § 16, p. 17, though written in 1896, makes no provision for nor mention of G_{360} discovered at least seven years before.

In the Kansas University Quarterly of January 1901, H. B. Newson presents "The Group of 216 Collineations in the Plane," from the one to eighteen correspondence with the tetrahedral group. This paper was not known to me when my note was made in the J. H. U. Circular of Jan. 1905.

18. Relative to the Hesse group the purpose is to derive its collineations by purely geometrical processes from certain conics.

Theorem. Three point-conics such that each is apolar to the remaining two in line form give rise to and determine the syzygetic pencil of cubics in the Hesse canonical form.

The three conics

$$x_1^2 + 2\,m\,x_2x_3 = 0,\quad x_2^2 + 2\,m\,x_3x_1 = 0,\quad x_3^2 + 2\,m\,x_1x_2 = 0, \tag{16}$$

are of this sort; and it has been shown that in the net of conics determined by three arbitrary conics there are just four sets of three conics each of this type, and that those of each set are tangent at the vertices by twos to the sides of one of the four inflexional triangles of the cubic of the net of conics.

The Jacobian[1] of these forms (16) is the cubic curve, the locus of points

[1] C.-L.: I, p. 378.

whose polars as to the three conics meet in a point, which point is also on the cubic and has the former point as its correspondent in the same way as it is of that point. The cubic f is then

$$f \equiv \begin{vmatrix} x_1 & m\,x_3 & m\,x_2 \\ m\,x_3 & x_2 & m\,x_1 \\ m\,x_2 & m\,x_1 & x_3 \end{vmatrix} = 0,$$

or $$f \equiv -\,m^2\,(x_1^3 + x_2^3 + x_3^3) + (1 + 2\,m^3)\,x_1 x_2 x_3 = 0, \qquad (3)$$

which is the Hessian of the cubic (2) § 1 whose parameter is $6\,m$, and so is a member of the syzygetic pencil given by equation (2) for all positive and negative values of m from 0 to ∞. Therefore, by varying the parameter in equations (16) we have three similar pencils of conics for the vertices two and two which give rise to the syzygetic pencil of cubics as above stated.

19. The Thirty-six Conics.

The inflexional triangles of this pencil are given by equations (4) and the four equianharmonic cubics have equations (5) p. 7 to define them.

The thirty-six conics are the first polars of the vertices of the inflexional triangles as to the four equianharmonic cubics, less twelve degenerate conics which are the squares of the sides severally of the inflexional triangles.

These twelve come from taking polars as indicated by $A_i\,S_1$, $B_i\,S_2$, $C_i\,S_3$, $D_i\,S_4$, $(i = 1, 2, 3)$. The subscript i indicates the vertex opposite the sides in order as given in equations (4), the letters show the respective triangles and equianharmonic cubics.

The conics in this way are as follows:

B_1S_1: $x_1^2 + x_2^2 + x_3^2 = 0.$ C_1S_1: $\omega\,x_1^2 + x_2^2 + x_3^2 = 0.$ D_1S_1: $\omega^2 x_1^2 + x_2^2 + x_3^2 = 0.$

B_2S_1: $x_1^2 + \omega\,x_2^2 + \omega^2 x_3^2 = 0.$ C_2S_1: $x_1^2 + \omega\,x_2^2 + x_3^2 = 0.$ D_2S_1: $x_1^2 + \omega^2 x_2^2 + x_3^2 = 0.$

B_3S_1: $x_1^2 + \omega^2 x_2^2 + \omega\,x_3^2 = 0.$ C_3S_1: $x_1^2 + x_2^2 + \omega\,x_3^2 = 0.$ D_3S_1: $x_1^2 + x_2^2 + \omega^2 x_3^2 = 0.$

C_1S_2: $\omega\,x_1^2 + x_2^2 + x_3^2 + 2\,(\omega\,x_2x_3 + x_3x_1 + x_1x_2) = 0.$

C_2S_2: $x_1^2 + \omega\,x_2^2 + x_3^2 + 2\,(x_2x_3 + \omega\,x_3x_1 + x_1x_2) = 0.$

C_3S_2: $x_1^2 + x_2^2 + \omega\,x_3^2 + 2\,(x_2x_3 + x_3x_1 + \omega\,x_1x_2) = 0.$

D_1S_2: $\omega^2 x_1^2 + x_2^2 + x_3^2 + 2\,(\omega^2 x_2x_3 + x_3x_1 + x_1x_2) = 0.$

D_2S_2: $x_1^2 + \omega^2 x_2^2 + x_3^2 + 2\,(x_2x_3 + \omega^2 x_3x_1 + x_1x_2) = 0.$

D_3S_2: $x_1^2 + x_2^2 + \omega^2 x_3^2 + 2\,(x_2x_3 + x_3x_1 + \omega^2 x_1x_2) = 0.$

A_1S_2: $x_1^2 + 2\,x_2x_3 = 0.$

A_2S_2: $x_2^2 + 2\,x_3x_1 = 0.$

A_3S_2: $x_3^2 + 2\,x_1x_2 = 0.$

$$D_1 S_3: \ \omega^2 x_1^2 + x_2^2 + x_3^2 + 2\,\omega^2 (\omega^2 x_2 x_3 + x_3 x_1 + x_1 x_2) = 0.$$
$$D_2 S_3: \ x_1^2 + \omega^2 x_2^2 + x_3^2 + 2\,\omega^2 (x_2 x_3 + \omega^2 x_3 x_1 + x_1 x_2) = 0.$$
$$D_3 S_3: \ x_1^2 + x_2^2 + \omega^2 x_3^2 + 2\,\omega^2 (x_2 x_3 + x_3 x_1 + \omega^2 x_1 x_2) = 0.$$
$$A_1 S_3: \ x_1^2 + 2\,\omega^2 x_2 x_3 = 0.$$
$$A_2 S_3: \ x_2^2 + 2\,\omega^2 x_3 x_1 = 0.$$
$$A_3 S_3: \ x_3^2 + 2\,\omega^2 x_1 x_2 = 0.$$
$$B_1 S_3: \ x_1^2 + x_2^2 + x_3^2 + 2\,\omega^2 (x_2 x_3 + x_3 x_1 + x_1 x_2) = 0.$$
$$B_2 S_3: \ x_1^2 + \omega x_2^2 + \omega^2 x_3^2 + 2\,\omega^2 (x_2 x_3 + \omega x_3 x_1 + \omega^2 x_1 x_2) = 0.$$
$$B_3 S_3: \ x_1^2 + \omega^2 x_2^2 + \omega x_3^2 + 2\,\omega^2 (x_2 x_3 + \omega^2 x_3 x_1 + \omega x_1 x_2) = 0.$$
$$A_1 S_4: \ x_1^2 + 2\,\omega x_2 x_3 = 0.$$
$$A_2 S_4: \ x_2^2 + 2\,\omega x_3 x_1 = 0.$$
$$A_3 S_4: \ x_3^2 + 2\,\omega x_1 x_2 = 0.$$
$$B_1 S_4: \ x_1^2 + x_2^2 + x_3^2 + 2\,\omega (x_2 x_3 + x_3 x_1 + x_1 x_2) = 0.$$
$$B_2 S_4: \ x_1^2 + \omega x_2^2 + \omega^2 x_3^2 + 2\,\omega (x_2 x_3 + \omega x_3 x_1 + \omega^2 x_1 x_2) = 0.$$
$$B_3 S_4: \ x_1^2 + \omega^2 x_2^2 + \omega x_3^2 + 2\,\omega (x_2 x_3 + \omega^2 x_3 x_1 + \omega x_1 x_2) = 0.$$
$$C_1 S_4: \ \omega x_1^2 + x_2^2 + x_3^2 + 2\,\omega (\omega x_2 x_3 + x_3 x_1 + x_1 x_2) = 0.$$
$$C_2 S_4: \ x_1^2 + \omega x_2^2 + x_3^2 + 2\,\omega (x_2 x_3 + \omega x_3 x_1 + x_1 x_2) = 0.$$
$$C_3 S_4: \ x_1^2 + x_2^2 + \omega x_3^2 + 2\,\omega (x_2 x_3 + x_3 x_1 + \omega x_1 x_2) = 0.$$

These conics form a closed system as explained and are in the same order except $B_2 S_3$ and $B_3 S_3$ as if derived as in § 14. Further, the line-forms are given by interchanging ω and ω^2 and writing ξ_i for x_i. We name point-conics with Roman caps; the same letters in script name the corresponding line-conics.

20. The Conics as Source of the 216 Collineations.

The product of a point-conic on a line-conic is a collineation. With this in mind we form a multiplication table with the point-conics along the left side and the line-conics along the top. The 216 collineations are written in matrix form and are numbered. The number is put in the square on the table corresponding with the two conics which produce the collineation of that number. Thus arranged the collineations were readily classified, from their actions on the inflexional triangles (and for simplicity on triangle A) into the tetrahedral subgroups.

In the multiplication of matrices remember that the separate terms of the rows upon the terms of the columns of multiplicand give rows of product.

In numbering I put a/b, where a is the number of the collineation given and b is that of the one having ω and ω^2 interchanged.

21. The 216 Collineations.

1	2/3	4	5	6/9	7/8
$\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{vmatrix}$	$\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\\omega^2&0&0\\0&\omega&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\0&0&\omega\\\omega^2&0&0\end{vmatrix}$

10	11	12	13/16	14/17	15/18
$\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&0\\0&0&\omega\\0&\omega^2&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\0&\omega&0\\\omega^2&0&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\\omega&0&0\\0&0&\omega^2\end{vmatrix}$

19/28	20/29	21/30	22/34	23/35	24/36
$\begin{vmatrix}1&1&1\\1&\omega^2&\omega\\1&\omega&\omega^2\end{vmatrix}$	$\begin{vmatrix}\omega^2&1&\omega\\1&1&1\\\omega&1&\omega^2\end{vmatrix}$	$\begin{vmatrix}\omega^2&\omega&1\\\omega&\omega^2&1\\1&1&1\end{vmatrix}$	$\begin{vmatrix}1&1&1\\\omega&1&\omega^2\\\omega^2&1&\omega\end{vmatrix}$	$\begin{vmatrix}\omega&\omega^2&1\\1&1&1\\\omega^2&\omega&1\end{vmatrix}$	$\begin{vmatrix}1&\omega^2&\omega\\1&\omega&\omega^2\\1&1&1\end{vmatrix}$

25/31	26/32	27/33
$\begin{vmatrix}1&1&1\\\omega^2&\omega&1\\\omega&\omega^2&1\end{vmatrix}$	$\begin{vmatrix}1&\omega&\omega^2\\1&1&1\\1&\omega^2&\omega\end{vmatrix}$	$\begin{vmatrix}\omega&1&\omega^2\\\omega^2&1&\omega\\1&1&1\end{vmatrix}$

37/55	38/56	39/57	40/58	41/59	42/60
$\begin{vmatrix}\omega^2&1&1\\\omega&1&\omega\\\omega&\omega&1\end{vmatrix}$	$\begin{vmatrix}\omega&\omega&1\\\omega^2&1&1\\\omega&1&\omega\end{vmatrix}$	$\begin{vmatrix}\omega&1&\omega\\\omega&\omega&1\\\omega^2&1&1\end{vmatrix}$	$\begin{vmatrix}1&\omega&\omega\\1&\omega^2&1\\\omega&\omega&1\end{vmatrix}$	$\begin{vmatrix}\omega&\omega&1\\1&\omega&\omega\\1&\omega^2&1\end{vmatrix}$	$\begin{vmatrix}1&\omega^2&1\\\omega&\omega&1\\1&\omega&\omega\end{vmatrix}$

43/61	44/62	45/63	46/64	47/65	48/66
$\begin{vmatrix}1&\omega&\omega\\\omega&1&\omega\\1&1&\omega^2\end{vmatrix}$	$\begin{vmatrix}1&1&\omega^2\\1&\omega&\omega\\\omega&1&\omega\end{vmatrix}$	$\begin{vmatrix}\omega&1&\omega\\1&1&\omega^2\\1&\omega&\omega\end{vmatrix}$	$\begin{vmatrix}\omega^2&1&1\\\omega&\omega&1\\\omega&1&\omega\end{vmatrix}$	$\begin{vmatrix}1&1&\omega^2\\\omega&1&\omega\\1&\omega&\omega\end{vmatrix}$	$\begin{vmatrix}1&\omega^2&1\\1&\omega&\omega\\\omega&\omega&1\end{vmatrix}$

49/67	50/68	51/69	52/70	53/71	54/72
$\begin{vmatrix}1&\omega&\omega\\1&1&\omega^2\\\omega&1&\omega\end{vmatrix}$	$\begin{vmatrix}\omega&\omega&1\\1&\omega^2&1\\1&\omega&\omega\end{vmatrix}$	$\begin{vmatrix}1&\omega^2&1\\\omega&\omega^2&\omega^2\\1&1&\omega^2\end{vmatrix}$	$\begin{vmatrix}\omega^2&1&1\\1&1&\omega^2\\\omega^2&\omega&\omega^2\end{vmatrix}$	$\begin{vmatrix}1&1&\omega^2\\1&\omega^2&1\\\omega&\omega^2&\omega^2\end{vmatrix}$	$\begin{vmatrix}1&\omega^2&1\\\omega^2&1&1\\\omega^2&\omega^2&\omega\end{vmatrix}$

73/91	74/92	75/93	76/97	77/98	78/99
$\begin{vmatrix}\omega^2&0&0\\0&0&1\\0&1&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\0&\omega^2&0\\1&0&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\1&0&0\\0&0&\omega^2\end{vmatrix}$	$\begin{vmatrix}1&0&0\\0&0&\omega^2\\0&1&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\0&1&0\\\omega^2&0&0\end{vmatrix}$	$\begin{vmatrix}0&\omega^2&0\\1&0&0\\0&0&1\end{vmatrix}$

79/94	80/95	81/96	82/100	83/101	84/102
$\begin{vmatrix}1&0&0\\0&0&1\\0&\omega^2&0\end{vmatrix}$	$\begin{vmatrix}0&0&\omega^2\\0&1&0\\1&0&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\\omega^2&0&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}\omega^2&0&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&0\\0&\omega^2&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&0\\0&1&0\\0&0&\omega^2\end{vmatrix}$

85/103	86/104	87/105	88/106	89/107	90/108
$\begin{vmatrix}0&0&1\\1&0&0\\0&\omega^2&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\0&0&\omega^2\\1&0&0\end{vmatrix}$	$\begin{vmatrix}0&1&0\\0&0&1\\\omega^2&0&0\end{vmatrix}$	$\begin{vmatrix}0&0&\omega^2\\1&0&0\\0&1&0\end{vmatrix}$	$\begin{vmatrix}0&0&1\\\omega^2&0&0\\0&1&0\end{vmatrix}$	$\begin{vmatrix}0&\omega^2&0\\0&0&1\\1&0&0\end{vmatrix}$

Line-Conics / Point-Conics	$\mathfrak{B}_1\mathfrak{S}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{C}_1\mathfrak{S}_2$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{D}_1\mathfrak{S}_3$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{A}_1\mathfrak{S}_4$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	Line / Point
B_1S_1	1	3	2	82	83	84	100	101	102	109	110	111	127	128	129	10	11	12	19	20	21	73	74	75	118	119	120	91	92	93	136	138	137	28	29	30	B_1S_1
B_2S_1	2	1	3	83	84	82	102	100	101	112	113	114	130	131	132	13	14	15	22	23	24	76	77	78	121	122	123	94	95	96	139	140	141	31	32	33	B_2
B_3S_1	3	2	1	84	82	83	101	102	100	115	116	117	133	134	135	16	17	18	25	26	27	79	80	81	124	125	126	97	98	99	142	143	144	34	35	36	B_3
C_1S_1	100	101	102	1	2	3	82	84	83	199	200	201	64	65	66	91	95	99	190	191	192	10	14	18	208	209	210	73	77	81	55	56	57	181	182	183	C_1
C_2S_1	101	102	100	3	1	2	84	83	82	202	203	204	67	68	69	97	92	96	193	194	195	16	11	15	211	212	213	79	74	78	58	59	60	184	185	186	C_2
C_3S_1	102	100	101	2	3	1	83	82	84	205	206	207	70	71	72	94	98	93	196	197	198	13	17	12	214	215	216	76	80	75	61	62	63	187	188	189	C_3
D_1S_1	82	84	83	100	102	101	1	3	2	46	47	48	172	173	174	73	76	79	145	146	147	91	94	97	37	38	39	10	13	16	163	164	165	154	155	156	D_1
D_2S_1	83	82	84	102	101	100	2	1	3	49	50	51	175	176	177	80	74	77	148	149	150	98	92	95	40	41	42	17	11	14	166	167	168	157	158	159	D_2
D_3S_1	84	83	82	101	100	102	3	2	1	52	53	54	178	179	180	78	81	75	151	152	153	96	99	93	43	44	45	15	18	12	169	170	171	160	161	162	D_3
C_1S_2	127	132	134	181	186	188	37	41	45	1	5	4	118	125	123	136	140	144	208	209	210	190	192	191	10	18	14	46	52	49	109	112	115	199	205	202	C_1S_2
C_2S_2	128	130	135	183	185	187	39	40	44	4	1	5	123	118	125	144	136	140	213	211	212	195	194	193	11	16	15	53	50	47	110	113	116	206	203	200	C_2
C_3S_2	129	131	133	182	184	189	38	42	43	5	4	1	125	123	118	140	144	136	215	216	214	197	196	198	12	17	13	57	48	54	111	114	117	204	201	207	C_3
D_1S_2	109	113	117	55	59	63	145	150	152	136	140	144	1	5	4	118	125	123	172	178	175	64	70	67	127	133	130	154	156	155	10	15	17	163	164	165	D_1
D_2S_2	110	114	115	57	58	62	147	149	151	144	136	140	4	1	5	123	118	125	179	176	173	71	68	65	128	134	131	159	158	157	11	13	18	168	166	167	D_2
D_3S_2	111	112	116	56	60	61	146	148	153	140	144	136	5	4	1	125	123	118	177	174	180	69	66	72	129	135	132	161	160	162	12	14	16	170	171	169	D_3
A_1S_2	10	13	16	73	79	76	91	94	97	118	125	123	136	140	144	1	5	4	28	33	35	82	85	86	109	116	114	100	103	104	127	131	135	19	27	23	A_1
A_2S_2	11	14	17	77	74	80	98	92	95	123	118	125	144	136	140	4	1	5	36	29	31	87	83	88	110	117	112	105	101	106	128	132	133	24	20	25	A_2
A_3S_2	12	15	18	81	78	75	96	99	93	125	123	118	140	144	136	5	4	1	32	34	30	89	90	84	111	115	113	107	108	102	129	130	134	26	22	21	A_3
D_1S_3	28	32	36	208	213	215	172	177	179	190	193	196	145	151	148	19	25	22	1	7	6	199	207	203	181	189	185	163	165	164	154	155	156	10	15	17	D_1S_3
D_2S_3	29	33	34	209	211	216	174	176	178	191	194	197	152	149	146	23	20	26	6	1	7	207	203	199	185	181	189	167	166	168	158	159	157	18	11	13	D_2
D_3S_3	30	31	35	210	212	214	173	175	180	192	195	198	150	147	153	27	24	21	7	6	1	203	199	207	189	185	181	171	170	169	162	160	161	14	16	12	D_3
A_1S_3	91	94	97	10	16	13	73	76	79	208	212	216	55	62	60	100	106	108	181	189	185	1	6	7	199	203	207	82	89	87	64	71	69	190	195	197	A_1
A_2S_3	92	95	98	14	11	17	80	74	77	210	211	215	63	58	56	104	111	107	189	185	181	7	1	6	203	207	199	90	83	85	68	66	70	192	194	196	A_2
A_3S_3	93	96	99	18	15	12	78	81	75	209	213	214	59	57	61	103	105	102	185	181	189	6	7	1	207	199	203	88	86	84	72	67	65	191	193	198	A_3
B_1S_3	136	141	143	190	194	198	46	50	54	10	11	12	109	110	111	127	128	129	199	203	207	181	185	189	1	7	6	37	40	43	118	124	121	208	211	214	B_1
B_2S_3	137	139	144	191	195	196	48	49	53	18	16	17	117	115	116	135	133	134	207	199	203	185	189	181	6	1	7	44	38	41	122	119	125	209	212	215	B_2
B_3S_3	138	140	142	192	193	197	47	51	52	14	15	13	113	114	112	131	132	130	203	207	199	189	181	185	7	6	1	42	45	39	126	123	120	210	213	216	B_3
A_1S_4	73	76	79	91	97	94	10	13	16	37	44	42	163	167	171	82	88	90	154	159	161	100	107	105	46	49	52	1	9	8	172	176	180	145	153	149	A_1S_4
A_2S_4	74	77	80	95	92	98	17	11	14	45	40	38	165	166	170	86	83	89	156	158	160	108	101	103	53	48	50	8	1	9	176	180	172	153	149	145	A_2
A_3S_4	75	78	81	99	96	93	15	18	12	41	39	43	164	168	169	85	87	84	155	157	162	106	104	102	51	54	47	9	8	1	180	172	176	149	145	153	A_3
B_1S_4	118	122	126	64	68	72	154	158	162	127	128	129	10	11	12	109	110	111	163	166	169	55	58	61	136	139	142	145	149	153	1	8	9	172	176	180	B_1
B_2S_4	119	123	124	66	67	71	155	157	161	132	130	131	15	13	14	114	112	113	164	167	170	62	56	59	143	138	140	149	153	145	9	1	8	180	172	176	B_2
B_3S_4	120	121	125	65	69	70	156	159	160	134	135	133	17	18	16	116	117	115	165	168	171	60	63	57	141	144	137	153	145	149	8	9	1	176	180	172	B_3
C_1S_4	19	24	25	199	204	206	163	168	170	181	187	184	154	157	160	28	31	34	10	18	14	208	210	209	190	191	192	172	180	176	145	153	149	1	8	9	C_1
C_2S_4	20	22	26	201	203	205	164	166	171	188	185	182	155	158	161	35	29	32	15	11	16	212	211	213	194	195	193	180	176	172	149	145	153	9	1	8	C_2
C_3S_4	21	23	27	200	202	207	165	167	169	186	183	189	156	159	162	33	36	30	17	13	12	216	215	214	198	196	197	176	172	180	153	149	145	8	9	1	C_3
Point / Line	$\mathfrak{B}_1\mathfrak{S}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{C}_1\mathfrak{S}_2$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{D}_1\mathfrak{S}_3$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{A}_1\mathfrak{S}_4$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{B}_1$	$\mathfrak{B}_2$	$\mathfrak{B}_3$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	Point / Line

Multiplication Table of Point- on Line-Conics giving the collineations according to number.

109/127	110/128	111/129	112/133	113/134	114/135
$\begin{vmatrix} \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \\ \omega^2 & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & 1 \\ \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & \omega \\ \omega^2 & \omega^2 & \omega \\ 1 & \omega^2 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ \omega & 1 & \omega \\ \omega & \omega^2 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega^2 & \omega^2 & \omega \end{vmatrix}$
115/130	**116/131**	**117/132**	**118/136**	**119/138**	**120/137**
$\begin{vmatrix} 1 & \omega & \omega \\ 1 & 1 & \omega^2 \\ \omega^2 & \omega & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ 1 & \omega & \omega \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & 1 \\ \omega & \omega^2 & \omega^2 \\ \omega & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega^2 & \omega & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & 1 \\ \omega^2 & \omega^2 & \omega \\ 1 & \omega & \omega \end{vmatrix}$
121/142	**122/143**	**123/144**	**124/139**	**125/140**	**126/141**
$\begin{vmatrix} 1 & \omega & \omega \\ \omega^2 & \omega & \omega^2 \\ 1 & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ \omega & \omega^2 & \omega^2 \\ \omega & 1 & \omega \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & \omega \\ 1 & \omega^2 & 1 \\ \omega^2 & \omega^2 & \omega \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & 1 \\ \omega & \omega & 1 \\ \omega & \omega^2 & \omega^2 \end{vmatrix}$

145/181	146/182	147/183	148/184	149/185	150/186
$\begin{vmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix}$	$\begin{vmatrix} \omega & \omega^2 & 1 \\ 1 & 1 & 1 \\ \omega & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & 1 \\ \omega^2 & \omega & 1 \\ 1 & \omega & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} \omega^2 & \omega & 1 \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{vmatrix}$
151/187	**152/188**	**153/189**	**154/190**	**155/191**	**156/192**
$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ \omega^2 & 1 & \omega \end{vmatrix}$	$\begin{vmatrix} \omega^2 & 1 & \omega \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & \omega \\ \omega^2 & 1 & \omega \\ 1 & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & \omega \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \\ 1 & 1 & 1 \end{vmatrix}$
157/193	**158/194**	**159/195**	**160/196**	**161/197**	**162/198**
$\begin{vmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega \end{vmatrix}$	$\begin{vmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ 1 & \omega & \omega^2 \end{vmatrix}$	$\begin{vmatrix} \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \\ 1 & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} \omega & 1 & \omega^2 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{vmatrix}$	$\begin{vmatrix} \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{vmatrix}$

163/208	164/209	165/210	166/211	167/212	168/213
$\begin{vmatrix} 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ \omega^2 & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \\ \omega & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} \omega & 1 & 1 \\ \omega^2 & 1 & \omega^2 \\ 1 & 1 & \omega \end{vmatrix}$	$\begin{vmatrix} \omega^2 & \omega^2 & 1 \\ \omega & 1 & 1 \\ 1 & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ 1 & \omega^2 & \omega^2 \end{vmatrix}$
169/214	**170/215**	**171/216**	**172/199**	**173/200**	**174/201**
$\begin{vmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ \omega^2 & \omega^2 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega \\ 1 & \omega^2 & \omega^2 \\ 1 & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} \omega^2 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega^2 & \omega^2 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} \omega^2 & \omega^2 & 1 \\ 1 & \omega & 1 \\ \omega & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} \omega^2 & 1 & \omega^2 \\ \omega & 1 & 1 \\ 1 & 1 & \omega \end{vmatrix}$
175/202	**176/203**	**177/204**	**178/205**	**179/206**	**180/207**
$\begin{vmatrix} \omega & 1 & 1 \\ \omega^2 & \omega^2 & 1 \\ 1 & \omega & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega \\ \omega^2 & 1 & \omega^2 \\ \omega & 1 & 1 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & 1 \\ 1 & \omega^2 & \omega^2 \\ 1 & 1 & \omega \end{vmatrix}$	$\begin{vmatrix} \omega & 1 & 1 \\ 1 & 1 & \omega \\ \omega^2 & 1 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & 1 \\ 1 & \omega^2 & \omega^2 \end{vmatrix}$	$\begin{vmatrix} 1 & \omega & 1 \\ \omega & 1 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}$

22. From the system of forms for the G_{216} as given by Maschke (Math. Ann. 33:324) we see that all the collineations can be deduced from six particular ones of the conics in line-form with two of these in point-form also. These six are

$A_1 S_1$, $A_2 S_1$, $A_3 S_1$, $A_1 S_2$, $A_2 S_2$, and $C_1 S_4$; with the point-conics $A_1 S_1$ and $A_1 S_2$. All the 216 collineations can be deduced from these conics and no other collineations arise therefrom.

THE TETRAHEDRAL SUB-GROUPS.

23. The sub-groups arising from the isomorphism with the G_{12} in the plane are these:

to the identical transformation corresponds				1 G_{18},
" " 3 sub-groups	G_2	correspond		3 G_{36}'s,
" " 4 " "	G_3	"		4 G_{54}'s,
" " 1 " "	G_4	corresponds		1 G_{72}.

The four flex-triangles are regarded as the four faces of the regular tetrahedron. The collineations naturally then transform the four triangles thus:

The triangles		$\underline{ABCD}$	are sent respectively	
into	"	$ABCD$	by collineations	1–18,
"	"	$BADC$	" "	19–36,
"	"	$CDAB$	" "	37–54,
"	"	$DCBA$	" "	55–72.

The first eighteen form the identity G_{18}; the other three sets of eighteen each, each with the collineations of the G_{18}, form the three G_{36}'s which interchange the triangles by twos. All these form the one G_{72}, which permutes all four triangles.

Next the triangles are permuted as follows:

triangles	$\underline{ABCD}$	are sent respectively into triangles	
1.	$ADBC$	by collineations numbered	73–90,
	$ACDB$	" " "	91–108,
2.	$DBAC$	" " "	109–126,
	$CBDA$	" " "	127–144,
3.	$DACB$	" " "	145–162,
	$BDCA$	" " "	163–180,
4.	$CABD$	" " "	181–198,
	$BCAD$	" " "	199–216.

The collineations of each of these four sets in connection with those of the G_{18} form a sub-group G_{54}, leaving one triangle unaltered and permuting the other three cyclically.

The collineations of the G_{18} of course send each respective inflexional triangle into itself, but not point for point in each case. A comparison of the G_{18} with respect to triangle A at once shows that, since A is, therefore, each triangle is sent by three collineations identically into itself. The collineations doing this are in each case identity and two of those of period three. Thus the action of identity and the eight of period three is accounted for. The remaining nine collineations of period two divide, differently for each triangle, into sets of three. Those of each set act the same on the triangle considered, sending one vertex into itself and interchanging the other two; that is, in other words, as is seen by comparing the equations of the harmonic polars, one of the three harmonic polars on the fixed vertex of the triangle remains, being the fixed line of the collineation, and the other two harmonic polars on that vertex are sent the one into the other.

Other Sub-Groups.

24. These sub-groups are those observed by noting the effect of the collineations on the 36 conics. Notice that the conics naturally divide into sets of nine, each set composed of three sets of three.

By examining the collineations of the G_{18} we readily see that numbers 1, 4, 5, 10, 11, 12 form a dihedral group G_6, sending conic $B_1 S_1$ into itself. Hence from symmetry, there are 12 dihedral G_6's, one for each set of three conics. These sub-groups each contain the identity collineation, two of period 3 and three of period 2; there is thus in each dihedral G_6 a cyclic G_3 which is an invariant sub-group. So the transform of one of the collineations of period two by one of the G_3 is one of those of period two.

The G_{18} contains identity, eight collineations of period 3 and nine of period 2. The former ones compose four cyclic G_3's; the latter with identity compose nine cyclic G_2's.

Next, we observe that the collineations of the three sets of 18 each which with the G_{18} form the G_{72}, are all of period four. These by sixes have a common second power. See Valentiner *loc. cit.* p. 69. These nine second powers are the nine collineations of period two. So one transformation of second order, the two of fourth order which produce it, and identity form a sub-group of period four. Thus there are twenty-seven cyclic G_4's. It seems strange that Valentiner should have overlooked and even practically denied the existence of the G_{216}, when he knew there must be 54 transformations of period four, and states that this number is $\frac{1}{4} N$, where N represents the total number of transformations.

He further gets the G_{72} with its sub-groups. I am inclined to question the complete accuracy of the French résumé for at the end of § 45, pp. 173, 227, the Danish and the French are exactly contradictory.

The cyclic G_4's divide the conics into sets of two and four, half of a set belonging to each of two natural sets of three.

In each of the eight sets of 18 each, which remain to be considered, 9 collineations are of period three and 9 of period six. These former of each of the four sets (1, 2, 3, 4, p. 25) of thirty-six with identity form 9 cyclic G_3's, or in all 36 more cyclic G_3's making the total number 40 cyclic G_3's. These G_3's separate the conics into sets of three and permute them in the set. Further, the collineations of period six in each set of 36 appear by twos in 9 cyclic G_6's. A cyclic G_6, different from a dihedral G_6, contains two collineations of period 6, two of period 3, one of period 2, and identity. There are thus from each set of 36 collineations above mentioned, together with the 9 of period two and identity each time, nine cyclic G_6's or in all 36 cyclic G_6's. These separate the conics into sets of nine. Two of the collineations of a G_6 send one of the nine conics into itself, two more send the same conic into each of the other two of its natural set of three, and the remaining two, which are of period six, send it into one of the remaining six conics of the set of nine.

Jordan says, *loc. cit.* p. 18, that a group G belonging to this (Hessian) type (of order $24 \cdot 9\phi$) contains a group H of order 27. "The substitutions of G are permutable with H." In each G_{54} there is such a group G_{27} consisting of the 26 collineations of period three and identity. So further we have 4 G_{27}'s.

Three of the collineations of period three from each lot of 36 above referred to, together with two of the eight of order three in G_{18}, and identity, form an Abelian G_9. Thus finally there are four Abelian G_9's.

25. To recapitulate, the Hesse group contains besides the sub-groups of § 23, these others:

9 cyclic G_2's, 40 cyclic G_3's, 27 cyclic G_4's, 12 dihedral G_6's, 36 cyclic G_6's, 4 Abelian G_9's and 4 G_{27}'s. These arrange themselves in interesting form on the multiplication table as the heavy rulings help to indicate.

The classification as to periodicity is

1	collineation identity,			
9	collineations	of	period	2,
80	"	"	"	3,
54	"	"	"	4,
72	"	"	"	6.

IV.

PERSPECTIVE TRIANGLES.—COMPLETE PAPPUS HEXAGON.

26. Having treated so fully the relations of the four inflexional triangles, which are in six-fold perspective, we should show how to obtain sets of triangles in the other perspective forms. Historically, it is of interest to note some papers. In 1870, H. Schröter (Math. Annalen 2 : 553–562) set for himself the question whether a triangle can be in more than single perspective with another triangle, and if so, in what other forms. His paper is characteristically clear and easy of reading, its method is synthetic, and it presents a construction for triangles in all possible forms; viz., one-, two-, three-, four-, and six-fold perspective.

J. Vályi, whose paper was referred to in § 16, p. 17, set for himself in 1882 the same problem and reached the same results analytically without reference to Schröter. Some other papers directly or indirectly presenting perspective triangles are simply noted :

Rosanes: Ueber Dreiecke in persp. Lage. Math. Ann. 2 : 549.

Hess: Beiträge z. Theorie d. mehrfach persp. Dreiecke. Ibid. 28 : 167.

Third: Triangles triply in Persp. Proc. Edinburg. Math. Soc. XIX, p. 10.

L. Klug: Desmische Vierseiten-Systeme. Monatshefte (1903) XIV, s. 74.

M. Pasch: Ueber Vier-eck und seit. Math. Ann. 26 : 211–216.

Caporali: Memorie, pp. 236, 252.

Veronese: Sull' Hexagrammum mysticum. Lincei Mem. II, 1 (1877), p. 649.

Triply Perspective Triangles in Circular Coördinates.

27. It is well known that two concentric equilateral triangles are triply perspective. We take two such, first as point-triads, with coördinates of vertices resp.,

$$\frac{1 \quad 2 \quad 3}{1 \quad \omega \quad \omega^2} \qquad \frac{1' \quad 2' \quad 3'}{at \quad \omega^2 at \quad \omega at} \tag{17}$$

These two point-triads are in perspective thus,

1	2	3	having Center,	and Axis of perspective
1′	2′	3′	α,	A,
2′	3′	1′	β,	B,
3′	1′	2′	γ,	Γ.

We name the vertices of the triangle of axes opposite correspondingly named axes A, B, C.

The line $\overline{11'}$ has coörd.	$1-\frac{a}{t}$,	$at-1$,	$\frac{a}{t}-at$,
" " $\overline{22'}$ " "	$\omega^2-\frac{\omega a}{t}$,	$\omega^2 at-\omega$,	$\frac{\omega^2 a}{t}-\omega at$,
" " $\overline{33'}$ " "	$\omega-\frac{\omega^2 a}{t}$,	$\omega at-\omega^2$,	$\frac{\omega a}{t}-\omega^2 at$.

The determinant of these coördinates vanishes identically for the sum of each column is zero. Therefore the lines meet in a point α. Similarly we show the points β and γ, and find then the coördinates of the three points to be

$$\alpha\equiv x=\frac{a\,(at^3-1)}{t\,(1-a^2)};\ \beta\equiv x=\frac{\omega a\,(at^3-1)}{t\,(1-a^2)};\ \gamma\equiv x=\frac{\omega^2 a\,(at^3-1)}{t\,(1-a^2)}. \quad \textbf{(18)}$$

From these coördinates we see that these three centers of perspective are on a circle concentric with the circles about the given triads, equally spaced and ordered as 1, 2, 3. The radius of their circle is

$$\frac{a}{1-a^2}\sqrt{\frac{(at^3-1)\,(a-t^3)}{t^3}}. \quad \textbf{(19)}$$

With radii of the given circumcircles resp. a and b this derived circle has radius

$$\frac{ab}{a^2-b^2}\sqrt{\frac{(at^3-b)\,(a-bt^3)}{t^3}}.$$

The coördinates of sides opposite the vertices 1, 2, 3 resp. are proportional to

$$(1,\ 1,\ 1),\qquad (1,\ \omega^2,\ \omega),\qquad (1,\ \omega,\ \omega^2). \quad \textbf{(20)}$$

Those of sides opposite $1'$, $2'$, $3'$, resp. are

$$(1,\ t^2,\ at),\qquad (1,\ \omega t^2,\ \omega^2 at),\qquad (1,\ \omega^2 t^2,\ \omega at). \quad \textbf{(21)}$$

Taking these sides in the same order of perspective as above we deduce the coördinates of the axes. From coördinates (20) and (21), we write off at once

$$x_{11'}=\frac{at-t^2}{t^2-1},\qquad x_{22'}=\frac{\omega at-\omega^2 t^2}{\omega t^2-\omega^2},\qquad x_{33'}=\frac{\omega^2 at-\omega t^2}{\omega^2 t^2-\omega},$$

where $x_{ii'}$ is the point of intersection of sides i and i'.

These three points lie on a line for we have identically

$$\begin{vmatrix} at-t^2 & 1-at & t^2-1 \\ \omega at-\omega^2 t^2 & \omega-\omega^2 at & \omega t^2-\omega^2 \\ \omega^2 at-\omega t^2 & \omega^2-\omega at & \omega^2 t^2-\omega \end{vmatrix}=0.$$

Thus we get the coördinates of axis

$$\begin{array}{llll} \text{A to be} & [at^3-1, & t(a-t^3), & t^2(1-a^2)]. \\ \text{B " "} & [at^3-1, & \omega^2 t(a-t^3), & \omega t^2(1-a^2)]. \\ \Gamma \text{ " "} & [at^3-1, & \omega t(a-t^3), & \omega^2 t^2(1-a^2)]. \end{array} \tag{22}$$

From these we get the coörds. of the vertices of this triangle of the axes thus,

$$\text{The meet of B}\Gamma \text{ is } A \equiv x = \frac{\begin{vmatrix} \omega^2 t(a-t^3) & \omega t^2(1-a^2) \\ \omega t(a-t^3) & \omega^2 t^2(1-a^2) \end{vmatrix}}{\begin{vmatrix} at^3-1 & \omega^2 t(a-t^3) \\ at^3-1 & \omega t(a-t^3) \end{vmatrix}} = \frac{(1-a^2)\,t^2}{at^3-1}.$$

$$\text{Similarly,} \qquad B \equiv x = \frac{\omega(1-a^2)\,t^2}{at^3-1}, \qquad C \equiv x = \frac{\omega^2(1-a^2)\,t^2}{at^3-1}. \tag{23}$$

Clearly these points form an equilateral triangle, concentric with the other three, and ordered as 1, 2, 3. The radius of its circumcircle is

$$(1-a^2)\sqrt{\frac{t^3}{(at^3-1)(a-t^3)}}. \tag{24}$$

The radii of the four circumcircles are seen to form the proportion,

$$1 : \frac{a}{1-a^2}\sqrt{\frac{(at^3-1)(a-t^3)}{t^3}} = (1-a^2)\sqrt{\frac{t^3}{(at^3-1)(a-t^3)}} : a;$$

or, in other words, the derived circles are a pair mutually inverse as to the same circle as to which the original circles are inverse. Therefore, we may state the theorem,

Two concentric equilateral triangles are in triple perspective with their centers of perspective and the three axes also equiangular triads concentric with the original two; and the radii of the circumcircles of the latter two triads are functions of the radii of the original circumcircles and of the clinant of the angle between the given triads.

The product of the radii of the latter two circles equals the product of those of the former two; hence, the circles are by pairs mutually inverse in the same circle.

The point-triads 1, 2, 3, and 1′, 2′, 3′, are each in triple perspective with α, β, γ, thus,

1	2	3	with perspective centers	1′	2′	3′	centers
α	γ	β	1′	α	β	γ	1
β	α	γ	2′	γ	α	β	2
γ	β	α	3′	β	γ	α	3

So, as is known, two point-triads in triple perspective with their triad of centers of perspective are a set of triads two and two in triple perspective with the points of the third triad as centers.

TRIANGLES 1, 2, 3 AND A, B, C.

28. These point-triads as here written are in anticyclic triple perspective with centers of perspective α', β', γ'. From coördinates (17) and (23) we write the coördinates of the perspecting rays:

$$\overline{1A}:\ \frac{a-t^3-(1-a^2)t}{a-t^3},\quad \frac{1-at^3+(1-a^2)t^2}{at^3-1},\quad \frac{t(1-a^2)(t^4+at^3-at-1)}{(a-t^3)(at^3-1)}.$$

$$\overline{2C}:\ \frac{\omega^2(a-t^3)-\omega(1-a^2)t}{a-t^3},\quad \frac{\omega(1-at^3)+\omega^2(1-a^2)t^2}{at^3-1},\quad \frac{\omega t(1-a^2)(t^4+\omega at-at-\omega)}{(a-t^3)(at^3-1)}.$$

$$\overline{3B}:\ \frac{\omega(a-t^3)-\omega^2(1-a^2)t}{a-t^3},\quad \frac{\omega^2(1-at^3)+\omega(1-a^2)t^2}{at^3-1},\quad \frac{\omega^2 t(1-a^2)(t^4+\omega^2 at-at-\omega^3)}{(a-t^3)(at^3-1)}.$$

The determinant of these coördinates vanishes identically, therefore the three lines meet in a point. From these and similar coördinates we find,

$$\alpha'\equiv x=\frac{(1-a^2)t}{a(at^3-1)},\quad \beta'\equiv x=\frac{\omega^2(1-a^2)t}{a(at^3-1)},\quad \gamma'\equiv x=\frac{\omega(1-a^2)t}{a(at^3-1)}. \tag{25}$$

The radius of their circumcircle is $\dfrac{1-a^2}{a}\sqrt{\dfrac{t^3}{(at^3-1)(a-t^3)}}$. (26)

The axes of perspective are found by taking the intersections of the sides thus,

1	2	3							
Α	Γ	Β	giving side of triangle 1′ 2′ 3′				opposite		1′.
Β	Α	Γ	"	"	"	"	"	"	2′.
Γ	Β	Α	"	"	"	"	"	"	3′.

By the same steps we show that

TRIANGLES 1′, 2′, 3′, AND A, B, C,

are in cyclic triple perspective with centers of perspective α'', β'', γ'', and axes of perspective the sides of triangle 1, 2, 3 in order 1, 3, 2. The radius of the circumcircle of α'', β'', γ'' is

$$a(1-a^2)\sqrt{\frac{t^3}{(at^3-1)(a-t^3)}}. \tag{27}$$

From a comparison of the radii of $\alpha\beta\gamma$ (19), of $\alpha'\beta'\gamma'$ (26) and of $\alpha''\beta''\gamma''$ (27), we may summarize thus:

The circumcircle of α, β, γ, which are the perspective centers of $\begin{Bmatrix}1 & 2 & 3\\ 1' & 2' & 3'\end{Bmatrix}$, *and* *the circle of* α', β', γ', *the centers of* $\begin{Bmatrix} 1 & 2 & 3 \\ A & C & B \end{Bmatrix}$, *whose axes are* $1'$, $2'$, $3'$, *are inverse to the circle of* 1, 2, 3; *also the circle of* α, β, γ, *and that of* α'', β'', γ'', *the centers of* $\begin{Bmatrix} 1' & 2' & 3' \\ A & B & C \end{Bmatrix}$, *whose axes are* 1, 3, 2, *are inverse to the circle of* $1'$, $2'$, $3'$.

The reader can very readily draw the figure for all the steps above outlined, so we leave that to him.

29. As a more interesting source of triply perspective triangles, and one involving some considerations of the cubic, we present what we shall call

THE COMPLETE PAPPUS HEXAGON.

In Pappi Alexandrini Mathematicae Collectiones[1] *a Federico Commandino Urbinate* as Prop. 138, p. 368, we read:

Si parallelae sint *AB*, *CD*, atque in ipsas incidant quaedam rectae lineae *AD*, *AF*, *BC*, *BF*, & *ED*, *EC*, jungantur, rectam lineam esse, quae per *GMK* puncta transit; and as Prop. 139, p. 368:

Sed non sint *ABCD* parallelae, & in puncto *N* conveniant. Dico rursus rectam lineam esse, quae per *GMK* puncta transit.

These theorems Pappus proved by proportion, the equal ratios being respectively between two lines and between the rectangles of two pairs of lines.

Salmon[2] has the same theorem stated thus:

"If *ABC* are three points of one line and *A'B'C'* are three points of another line, then the intersections *BC'*/*B'C*, *CA'*/*C'A*, *AB'*/*A'B* lie on a line."

The most important mention of the simple case is by Rudolf Böger,[3] who gives it as a simple form, free from the perspective relations, of *Das Sechseck in der Geometrie der Lage.*

30. The complete figure is constructed thus:

The numbers 1, 3, 5, and 2, 4, 6, are regarded as the names of points or of lines, each set of three lying on a line or a point resp. We consider the cross-joins as follows:

1	2	3	4	5	6	(1)	(2)	(3)	[1]	on Σ_1 or D_1 resp.	(28)
3	2	5	4	1	6	(4)	(5)	(6)	[2]		
5	2	1	4	3	6	(7)	(8)	(9)	[3]		
3	2	1	4	5	6	(10)	(11)	(12)	[4]	on Σ_2 or D_2 resp.	(29)
5	2	3	4	1	6	(13)	(14)	(15)	[5]		
1	2	5	4	3	6	(16)	(17)	(18)	[6]		

[1] From U. S. Cong. Library; also by F. Hultsch in 3 Vols. giving the Greek text also, see Vol. 2, p. 885.

[2] Conic Sections (6th Ed.) §268, p. 246, Ex. 1.

[3] Sechseck und Involution, Mitteilungen d. Math. Ges. in Hamburg, Bd. III, Feb. 1899, s. 387.

This means, *when the six numbers are points*, that the line of $\overline{1, 2}$ intersects that of $\overline{4, 5}$ in the point (1); the line $\overline{2, 3}$ cuts line $\overline{5, 6}$ in point (2); and $\overline{3, 4}/\overline{6, 1}$ is the point (3); further, that point (1), (2), (3) lie on a line [1] which is one of three lines similarly got and lying on point Σ_1. Observe there are nine lines, such as $\overline{1, 2}$, connecting the two sets of three points each, which have not been named. In the dual figure we shall call the nine points α_{1-3}, β_{1-3}, γ_{1-3}.

On the figures, the six points shall be marked π_i and the six lines P_i; the points or lines (n) shall be marked simply n. For convenience in the analytical work slight changes are there made as to names but these will be readily followed on the figures throughout. Following the names given to points and lines in the Pascal hexagon, the points Σ_1 and Σ_2 are called Steiner points, whose line is S; the lines P_i are called Pappus lines; and we call the lines D_1 and D_2 Hessian diagonals, which intersect in δ.

31. As a convenient projection of the hexagon, we take the points π_2, π_4, π_6 on a line considered the axis of reals, and the points π_1, π_3, π_5 on the line at infinity so that the lines from the three points on the axis to these three are equispaced lines, parallel respectively

to	$x = ty$	$x = \omega ty$	$x = \omega^2 ty$
going resp. to	π_1	π_3	π_5
Lines on $\pi_2 \equiv a$:	$x = ty - a(t-1)$.	$x = \omega ty - a(\omega t - 1)$.	$x = \omega^2 ty - a(\omega^2 t - 1)$.
" " $\pi_4 \equiv b$:	$x = ty - b(t-1)$.	$x = \omega ty - b(\omega t - 1)$.	$x = \omega^2 ty - b(\omega^2 t - 1)$. (30)
" " $\pi_6 \equiv c$:	$x = ty - c(t-1)$.	$x = \omega ty - c(\omega t - 1)$.	$x = \omega^2 ty - c(\omega^2 t - 1)$.

The points (p) — $p = 1, 2, \ldots, 9$ — are in general thus,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x = \frac{a - b\omega - (a-b)\omega^{i+j} t}{1 - \omega};$$

and the points (q) — $q = 10, 11, \ldots, 18$ — of § 30 are thus,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x = \frac{a - b\omega^2 - (a-b)\omega^{i+j-1} t}{1 - \omega^2};$$

or points (q) are got from points (p) by interchanging b and c in the equation where $p = q - 9$.

In the above, a and b each permute for a, b, c, but a is never b; and i, j are each 1, 3, 5, but i is never j in any one equation. a, b, c are the general points π_2, π_4, π_6 along line D_2.

These 18 points in 6 sets of three each, as indicated in § 30, lie on six lines P_i. From the equations of points (p) and (q) we get the coördinates of the lines P.

For lines P_i, where $i=1, 3, 5$, we find coördinates are

$$P_i:\ a+b\omega+c\omega^2,\quad -(a+b\omega^2+c\omega)\omega^{-i}t,\quad \omega^2(ab+bc\omega+ca\omega^2)-(ab+bc\omega^2+ca\omega)\omega^{-i+1}t.$$

The coördinates of lines P_j, where $j=2, 4, 6$, are

$$P_j:\ a+b\omega^2+c\omega,\quad -(a+b\omega+c\omega^2)\omega^{-j+1}t,\quad \omega(ab+bc\omega^2+ca\omega)-(ab+bc\omega+ca\omega^2)\omega^{-j}t.$$

The three lines P_i lie on the point Σ_1, which is

$$x=-\omega^2\frac{ab+bc\omega+ca\omega^2}{a+b\omega+c\omega^2};$$

and the three lines P_j lie on the point Σ_2, which is

$$x=-\omega\frac{ab+bc\omega^2+ca\omega}{a+b\omega^2+c\omega}.$$

These two points are evidently conjugate and therefore symmetrical as to the axis of reals. Further, since the axis of reals, or D_2, bisects the line between Σ_1 and Σ_2, the two points Σ are harmonic as to the two lines D. They are the Hessian pair of the three points π_2, π_4, π_6.

Since the points Σ are independent of t, these points remain the same for any three equispaced lines on a, b, c resp., mutually parallel; or keeping one triad of points fixed the triad on the other line may move all along their line subject only to the condition that the angles between the lines on the fixed points remain constant. That is, Σ_1 and Σ_2 are the same for all triads on the second line having the same Hessian pair. Thus along the one line may be generated a pencil of triads with the same Hessian pair by turning the *equispaced* triad on Σ, each three points cut out at any instant being a triad of the pencil.

Reversing the Process.

32. Starting with the three lines P_i on Σ_1 and the three P_j on Σ_2, the lines P_{ij} intersect in nine points as follows:

$$P_1P_2:\ x=\frac{a^2-bc+(a-b)(a-c)\omega^2 t}{2a-b-c}.$$

$$P_1P_4:\ x=\frac{c^2-ab+(c-a)(c-b)\omega t}{2c-a-b}.$$

$$P_1P_6:\ x=\frac{b^2-ca+(b-c)(b-a)t}{2b-c-a}.$$

The remaining six may be written at once by comparing these three with the following table:

	P_1	P_3	P_5
P_2	a^2, ω^2	b^2, ω	$c^2, 1$
P_4	c^2, ω	$a^2, 1$	b^2, ω^2
P_6	$b^2, 1$	c^2, ω^2	a^2, ω

The lines joining these intersections in pairs as indicated in § 30 meet by threes in points along D_2 and D_1, which are not in general the original points on these lines.

The lines (q) to the three *new* points along D_1 are given by the second lot of three permutations (29) and have slopes respectively, t^2, ωt^2, $\omega^2 t^2$; so they turn just twice the angle from the axis as the original pair and are likewise equispaced.

The lines (p) intersect by threes on three new points, π'_1, π'_3, π'_5, along D_2, corresponding respectively with a, b, c, in order along the external segment of the line. They are,

$$x_a = \frac{\begin{vmatrix} a-b & b^2-ca \\ c-a & c^2-ab \end{vmatrix}}{\begin{vmatrix} a-b & 2b-c-a \\ c-a & 2c-a-b \end{vmatrix}}, \text{ where } a, b, c, \text{ permute cyclically.}$$

33. Thus far the origin on the axis has been arbitrary. Now consider it the centroid of the three given points, so that

$$a + b + c = 0,$$

whence also $$a^2 - bc = b^2 - ca = c^2 - ab = \lambda, \text{ say,}$$

and $$bc + ca + ab = -\lambda, \quad 2a - b - c = 3a, \text{ etc.}$$

The three new points, π'_i, then become resp.,

$$x = \frac{a\lambda}{-2bc + ca + ab}, \quad x = \frac{b\lambda}{bc - 2ca + ab}, \quad x = \frac{c\lambda}{bc + ca - 2ab}.$$

The counter-triad of the three points a, b, c, is

$$\frac{-2bc + ca + ab}{3a} \text{ from } (xa/bc) = -1; \text{ call it } a'.$$

$$\frac{bc - 2ca + ab}{3b} \text{ from } (xb/ca) = -1; \text{ call it } b'.$$

$$\frac{bc + ca - 2ab}{3c} \text{ from } (xc/ab) = -1; \text{ call it } c'.$$

The cubic along the line a, b, c is

$$x^3 - \lambda x - abc = 0.$$

Differentiating this as to x, we have

$$3x^2 - \lambda = 0,$$

the roots of which are the polar pair of infinity, the intersection of D_1 and D_2. Calling the roots f and f', we have

$$f + f' = 0, \text{ and } ff' = -f^2 = -f'^2 = -\frac{\lambda}{3}. \quad \therefore f^2 = f'^2 = \frac{\lambda}{3}.$$

Then by comparing the above values of this paragraph, we see

$$\pi_1' a' = \pi_3' b' = \pi_5' c' = \frac{\lambda}{3};$$

so the new triad and the counter-triad of the original triad are in an involution whose double-points are the polar pair of the intersection of the lines D_1 and D_2 as to the original triad.

Further, starting with the new triad as we did with the triad π_1, π_3, π_5, we shall secure a third triad π_1'', π_3'', π_5'', etc. continuously, all these triads having the same polar pair with respect to δ and the same Hessian pair.

Thus, there is constructed, as it were, a syzygetic pencil of triads of points along a line. For all the above there is of course its dual, giving a corresponding pencil of line triads on each of two points.

34. From the Complete Pappus Hexagon we have the following theorems with their duals:

I.

Three lines P_i on each of two points Σ_1, Σ_2, joined by the line S, intersect cross-wise in nine points α_{1-3}, β_{1-3}, γ_{1-3}, which join by 18 lines, which are the sides of two sets of three point-triads each.

The triangles of each set are *inter se* in triple perspective, having as centers of perspective the points Σ_1 and Σ_2 each three times, and three points π_i on the line D_2 or D_1 resp.; and having as axes of perspective the line D_1 or D_2 three times each for the sets resp., and twelve other lines all of which pass through a point ε, which is the pole of the line S as to any of the six triangles. The 18 lines above lie by three on six points π_i, which are three and three on the lines D_1 and D_2 above.

Three points π_i on each of two lines D_1, D_2, meeting in the point δ, are cross-joined by nine lines which meet in eighteen points, 1–18, which are the vertices of two sets of three line-triads each.

The triangles of each set are *inter se* in triple perspective, having as axes of perspective the lines D_1 and D_2 each three times, and three lines P_i on the point Σ_2 or Σ_1 respectively; and having as centers of perspective the point Σ_1 or Σ_2 three times each for the sets resp., and twelve other points all of which lie on a line E, which is the polar of the point δ as to any of the six triangles. The 18 points lie by three on six lines P_i which are three and three on the points Σ_1 and Σ_2 above.

II

The lines D_1, D_2 and S are the false sides of the complete quadrilateral of the Hessian pairs of the line-triads P_i on Σ_1 and on Σ_2.

The points Σ_1, Σ_2 and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point-triads π_i on D_1 and on D_2.

From these follows the general theorem, as also from the demonstration of §§ 31, 32.

III.

Three lines on each of two points give rise to three points on each of two lines, and the latter by reciprocating the process give rise to three lines on each of the original two points. The derived three lines have the same Hessian pair, or are inclined to each other at the same angle as the original three.

35. We now give a proof of the theorems on the left. Take the two Steiner points Σ_1, Σ_2, to have coördinates σ_1, σ_2, σ_3 and ς_1, ς_2, ς_3 resp.; the triangle $\alpha_1 \alpha_2 \alpha_3$, formed by the intersections of the Pappus lines as will be shown, as reference triangle; and the line S as auxiliary line.

The line S determined by Σ_1 and Σ_2 is given by

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \varsigma_1 & \varsigma_2 & \varsigma_3 \end{vmatrix} = 0, \text{ and will be written}$$

$$S_1 x_1 + S_2 x_2 + S_3 x_3 = 0.$$

Since this line is taken as auxiliary line

$$S_1 = S_2 = S_3 = 1, \quad \sigma_1 + \sigma_2 + \sigma_3 = 0, \quad \varsigma_1 + \varsigma_2 + \varsigma_3 = 0, \tag{31}$$

also
$$\varsigma_i^2 \sigma_j \sigma_k - \varsigma_j \varsigma_k \sigma_i^2 = \sigma_k \varsigma_k - \sigma_j \varsigma_j,$$

where i, j, k are each 1, 2, 3 successively.

The equations of the Pappus lines are the corresponding minors as represented thus,

P_1	P_3	P_5		P_2	P_4	P_6
σ_1	σ_2	σ_3		ς_1	ς_2	ς_3
x_1	x_2	x_3		x_1	x_2	x_3

Their respective intersections, corresponding with the nine points P_{ij} of § 32, have coördinates:

$$\begin{array}{lll} \alpha_1: (1, 0, 0), & \alpha_2: (0, 1, 0), & \alpha_3: (0, 0, 1), \\ \beta_1: (\sigma_1\varsigma_1, \sigma_1\varsigma_2, \sigma_3\varsigma_1), & \beta_2: (\sigma_1\varsigma_2, \sigma_2\varsigma_2, \sigma_2\varsigma_3), & \beta_3: (\sigma_3\varsigma_1, \sigma_2\varsigma_3, \sigma_3\varsigma_3), \\ \gamma_1: (\sigma_1\varsigma_1, \sigma_2\varsigma_1, \sigma_1\varsigma_3), & \gamma_2: (\sigma_2\varsigma_1, \sigma_2\varsigma_2, \sigma_3\varsigma_2), & \gamma_3: (\sigma_1\varsigma_3, \sigma_3\varsigma_2, \sigma_3\varsigma_3). \end{array} \tag{32}$$

where the intersections are named as indicated here:

Lines	P_3	P_1	P_5					
meet lines	P_6	P_2	P_4	resp. in points	α_1	α_2	α_3	
" "	P_2	P_4	P_6	" " "	β_3	β_1	β_2	**(33)**
" "	P_4	P_6	P_2	" " "	γ_2	γ_3	γ_1	

From coördinates (32) we write the coördinates of the joins:

$$\begin{array}{lll} \alpha_2\alpha_3: (1,\ 0,\ 0). & \alpha_3\alpha_1: (0,\ 1,\ 0). & \alpha_1\alpha_2: (0,\ 0,\ 1). \\ \beta_2\beta_3: (\sigma_2\varsigma_3,\ \sigma_3\varsigma_3,\ \sigma_2\varsigma_2). & \beta_3\beta_1: (\sigma_3\varsigma_3,\ \sigma_3\varsigma_1,\ \sigma_1\varsigma_1). & \beta_1\beta_2: (\sigma_2\varsigma_2,\ \sigma_1\varsigma_1,\ \sigma_1\varsigma_2). \\ \gamma_2\gamma_3: (\sigma_3\varsigma_2,\ \sigma_3\varsigma_3,\ \sigma_2\varsigma_2). & \gamma_3\gamma_1: (\sigma_3\varsigma_3,\ \sigma_1\varsigma_3,\ \sigma_1\varsigma_1). & \gamma_1\gamma_2: (\sigma_2\varsigma_2,\ \sigma_1\varsigma_1,\ \sigma_2\varsigma_1). \end{array} \qquad (34)$$

By observing that the determinant of the three sets of coördinates of each column above vanishes identically, after subtracting its third row from its second row and remembering (31), we see that the three lines of each column pass through a point. These three points are called π_1, π_3, π_5, and have coördinates resp.,

$$(0,\ -\sigma_2\varsigma_2,\ \sigma_3\varsigma_3),\quad (\sigma_1\varsigma_1,\ 0,\ -\sigma_3\varsigma_3),\quad (-\sigma_1\varsigma_1,\ \sigma_2\varsigma_2,\ 0);$$

and these three points clearly lie on a line, D_1, with coördinates,

$$(\sigma_2\sigma_3\varsigma_2\varsigma_3,\quad \sigma_3\sigma_1\varsigma_3\varsigma_1,\quad \sigma_1\sigma_2\varsigma_1\varsigma_2) \text{ or } \left(\frac{1}{\sigma_1\varsigma_1},\ \frac{1}{\sigma_2\varsigma_2},\ \frac{1}{\sigma_3\varsigma_3}\right). \qquad (35)$$

Again, from (32) we write the coördinates of the other set of joins thus:

$$\begin{array}{lll} \alpha_1\beta_1: (0,\ -\sigma_3\varsigma_1,\ \sigma_1\varsigma_2). & \beta_1\gamma_1: (\sigma_3\varsigma_3-\sigma_2\varsigma_2,\ -\sigma_1\varsigma_1,\ \sigma_1\varsigma_1). & \gamma_1\alpha_1: (0,\ -\sigma_1\varsigma_3,\ \sigma_2\varsigma_1). \\ \alpha_2\beta_2: (\sigma_2\varsigma_3,\ 0,\ -\sigma_1\varsigma_2). & \beta_2\gamma_2: (\sigma_2\varsigma_2,\ \sigma_1\varsigma_1-\sigma_3\varsigma_3,\ -\sigma_2\varsigma_2). & \gamma_2\alpha_2: (\sigma_3\varsigma_2,\ 0,\ -\sigma_2\varsigma_1). \\ \alpha_3\beta_3: (-\sigma_2\varsigma_3,\ \sigma_3\varsigma_1,\ 0). & \beta_3\gamma_3: (-\sigma_3\varsigma_3,\ \sigma_3\varsigma_3,\ \sigma_2\varsigma_2-\sigma_1\varsigma_1). & \gamma_3\alpha_3: (-\sigma_3\varsigma_2,\ \sigma_1\varsigma_3,\ 0). \end{array} \qquad (36)$$

Evidently the three lines of each column here also lie on a point, for the columns of determinants vanish identically on addition. The three points are called π_2, π_4, π_6 resp. and have coördinates,

$$\pi_2: (\sigma_3\varsigma_2,\ \sigma_1\varsigma_3,\ \sigma_2\varsigma_1),\quad \pi_4: (\sigma_2\varsigma_2+\sigma_3\varsigma_3-\sigma_1\varsigma_1,\ \sigma_3\varsigma_3+\sigma_1\varsigma_1-\sigma_2\varsigma_2,\ \sigma_1\varsigma_1+\sigma_2\varsigma_2-\sigma_3\varsigma_3),$$
$$\pi_6: (\sigma_2\varsigma_3,\ \sigma_3\varsigma_1,\ \sigma_1\varsigma_2).$$

These three points, on adding columns of their determinant, are seen to lie on the line D_2. Its coördinates are

$$\frac{\sigma_2\varsigma_2-\sigma_3\varsigma_3}{\sigma_1\varsigma_1},\qquad \frac{\sigma_3\varsigma_3-\sigma_1\varsigma_1}{\sigma_2\varsigma_2},\qquad \frac{\sigma_1\varsigma_1-\sigma_2\varsigma_2}{\sigma_3\varsigma_3}, \qquad (37)$$

The intersection of D_1 and D_2, viz., δ, has coördinates

$$[\sigma_1\varsigma_1(\sigma_2\varsigma_2+\sigma_3\varsigma_3-2\sigma_1\varsigma_1),\ \sigma_2\varsigma_2(\sigma_3\varsigma_3+\sigma_1\varsigma_1-2\sigma_2\varsigma_2),\ \sigma_3\varsigma_3(\sigma_1\varsigma_1+\sigma_2\varsigma_2-2\sigma_1\varsigma_1)]. \qquad (38)$$

The nine joins of each of the two sets, (34) and (36), are the sides of three triangles. These are resp. the point-triads

(1)	α_1	α_2	α_3	and	(1)	α_1	β_1	γ_1
(2)	β_1	β_2	β_3	"	(2)	α_2	β_2	γ_2
(3)	γ_1	γ_2	γ_3	"	(3)	α_3	β_3	γ_3

whose vertices as shown in (33) are, for each triangle, joins two and two of all six lines on Σ_1, Σ_2; and so the triangles are cubics passing through the same nine points.

The triangles of each set are mutually in triple perspective. The three centers of perspective are first π_2, π_4, π_6, as the coördinates of lines (34) show, and Σ_1, Σ_2 each three times as seen from the naming of points in (28) and (29). The centers for the second set are similarly π_1, π_3, π_5, and Σ_1, Σ_2 each three times.

The axes of perspective are clearly, once for each two triangles of the first set, the line D_1, and once for each two of the second set, the line D_2. This accounts for six of the eighteen axes. The remaining twelve are as follows:

$\alpha_1\alpha_2$ and $\beta_2\beta_3$ meet in point $(\sigma_3, -\sigma_2, 0)$.
$\alpha_2\alpha_3$ " $\beta_3\beta_1$ " " " $(0, \sigma_1, -\sigma_3)$.
$\alpha_3\alpha_1$ " $\beta_1\beta_2$ " " " $(-\sigma_1, 0, \sigma_2)$.

Whence, Axis I has coördinates $(\sigma_2, \sigma_3, \sigma_1)$,

for by subtracting second row of the determinant from the first row, and the third from the second, we have

$$\sigma_3\sigma_1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -\sigma_1 & 0 & \sigma_2 \end{vmatrix} \equiv 0,$$

$\alpha_1\alpha_2$ and $\beta_3\beta_1$ meet in point $(\varsigma_1, -\varsigma_3, 0)$.
$\alpha_2\alpha_3$ " $\beta_1\beta_2$ " " " $(0, \varsigma_2, -\varsigma_1)$.
$\alpha_3\alpha_1$ " $\beta_2\beta_3$ " " " $(-\varsigma_2, 0, \varsigma_3)$.

Whence Axis II has coördinates $(\varsigma_3, \varsigma_1, \varsigma_2)$.

Thus in order as indicated, we have the triply perspective triangles:

Triangles.	*Centers.*	*Axes of Perspective.*
$\alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\beta_3$	π_2, Σ_2, Σ_1.	D_1, $(\sigma_2, \sigma_3, \sigma_1)$, $(\varsigma_3, \varsigma_1, \varsigma_2)$.
$\alpha_1\alpha_2\alpha_3$, $\gamma_1\gamma_2\gamma_3$	π_4, Σ_1, Σ_2.	D_1, $(\varsigma_2, \varsigma_3, \varsigma_1)$, $(\sigma_3, \sigma_1, \sigma_2)$.
$\beta_1\beta_2\beta_3$, $\gamma_1\gamma_2\gamma_3$	π_6, Σ_2, Σ_1.	D_1, $(\sigma_1, \sigma_2, \sigma_3)$, $(\varsigma_1, \varsigma_2, \varsigma_3)$.
$\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$	π_1, Σ_2, Σ_1.	D_2, $(\sigma_2, \sigma_1, \sigma_3)$, $(\varsigma_2, \varsigma_1, \varsigma_3)$.
$\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$	π_3, Σ_2, Σ_1.	D_2, $(\varsigma_3, \varsigma_2, \varsigma_1)$, $(\sigma_3, \sigma_2, \sigma_1)$.
$\alpha_3\beta_3\gamma_3$, $\alpha_1\beta_1\gamma_1$	π_5, Σ_1, Σ_2.	D_2, $(\sigma_1, \sigma_3, \sigma_2)$, $(\varsigma_1, \varsigma_3, \varsigma_2)$.

The twelve axes other than D_1 and D_2 are seen to pass through the point with coördinates (1, 1, 1), which is thus the auxiliary point ε, the pole of S as to the reference triangle. Since

S has the same pole, ε, as to all point-cubics consisting of three joins of the six lines P_i, two and two,

ε is the pole of S as to every and any of the six triangles on all six lines P_i.

δ has the same polar, E, as to all line-cubics consisting of three joins of the six points π_i, two and two, (Salmon: H. P. C., § 166, pp. 143, 144),

E is the polar of δ as to every and any of the six triangles on all six points π_i.

The 18 lines have already been shown, (34) and (36), to lie by threes on the six points π_i which are on the lines D_1 and D_2 three and three.

THE LINES D_1 AND D_2 AS DIAGONALS OR FALSE SIDES OF THE COMPLETE QUADRILATERAL OF THE HESSIAN PAIRS.

36. The following linear relation exists as to the three lines P_i on Σ_1,

$$\sigma_1(\sigma_2 x_3 - \sigma_3 x_2) + \sigma_2(\sigma_3 x_1 - \sigma_1 x_3) + \sigma_3(\sigma_1 x_2 - \sigma_2 x_1) = 0.$$

The Hessian covariant of the binary cubic is given by the sum of the squares of these terms separately, and the imaginary Hessian lines are

$$\sigma_1(\sigma_2 x_3 - \sigma_3 x_2) + \omega\sigma_2(\sigma_3 x_1 - \sigma_1 x_3) + \omega^2\sigma_3(\sigma_1 x_2 - \sigma_2 x_1) = 0,$$

and

$$\sigma_1(\sigma_2 x_3 - \sigma_3 x_2) + \omega^2\sigma_2(\sigma_3 x_1 - \sigma_1 x_3) + \omega\sigma_3(\sigma_1 x_2 - \sigma_2 x_1) = 0.$$

These two and the analogous two on Σ_2 may be written in reduced form respectively

$$\sigma_2\sigma_3 x_1 + \omega\sigma_3\sigma_1 x_2 + \omega^2\sigma_1\sigma_2 x_3 = 0, \quad \{1\}$$
$$\sigma_2\sigma_3 x_1 + \omega^2\sigma_3\sigma_1 x_2 + \omega\sigma_1\sigma_2 x_3 = 0, \quad \{2\}$$
$$\varsigma_2\varsigma_3 x_1 + \omega\varsigma_3\varsigma_1 x_2 + \omega^2\varsigma_1\varsigma_2 x_3 = 0, \quad \{3\}$$
$$\varsigma_2\varsigma_3 x_1 + \omega^2\varsigma_3\varsigma_1 x_2 + \omega\varsigma_1\varsigma_2 x_3 = 0, \quad \{4\}$$

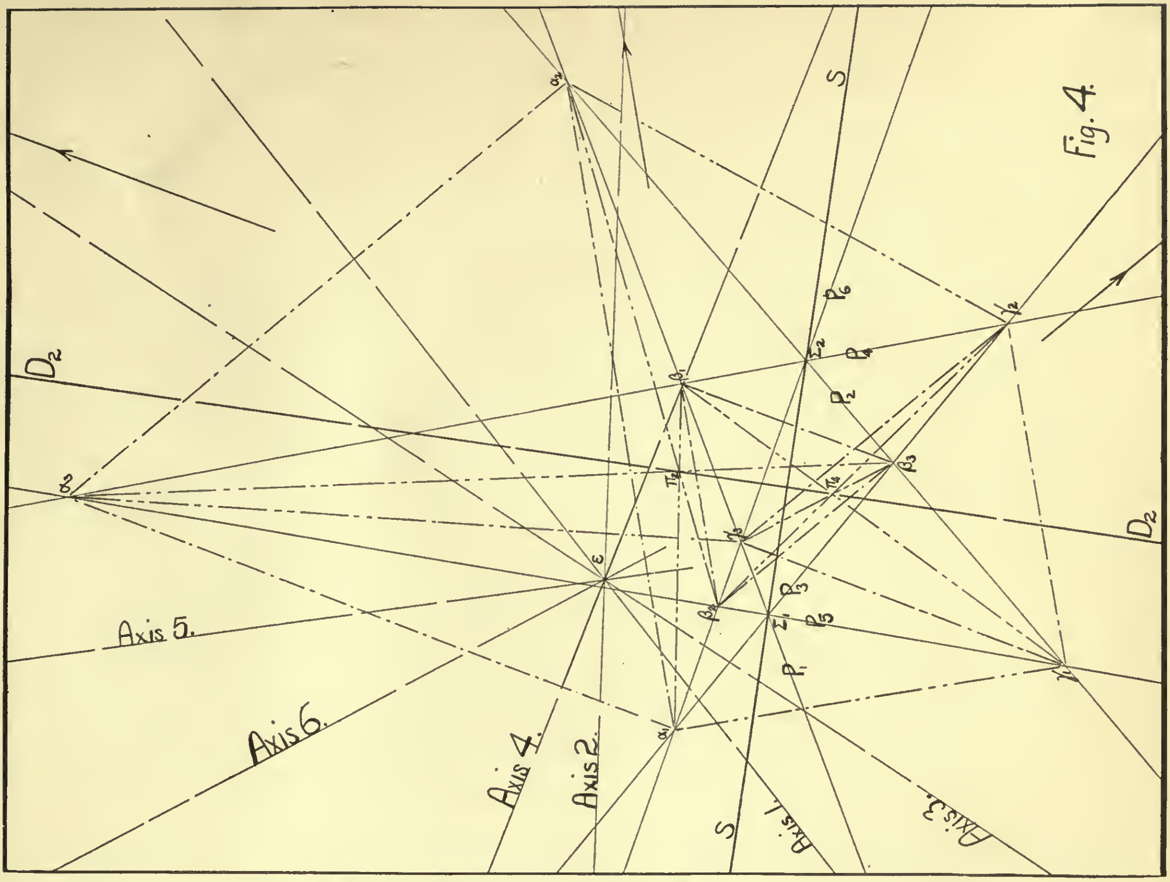
These intersect as follows:

{1} and {3} in imaginary point I: $(\sigma_1\varsigma_1,\ \omega^2\sigma_2\varsigma_2,\ \omega\sigma_3\varsigma_3)$.
{2} " {4} " " " J: $(\sigma_1\varsigma_1,\ \omega\sigma_2\varsigma_2,\ \omega^2\sigma_3\varsigma_3)$.
{1} " {4} " " " H: $(\sigma_1\varsigma_1[\sigma_2\varsigma_3 - \omega\sigma_3\varsigma_2],\ \sigma_2\varsigma_2[\sigma_3\varsigma_1 - \omega\sigma_1\varsigma_3],\ \sigma_3\varsigma_3[\sigma_1\varsigma_2 - \omega\sigma_2\varsigma_1])$.
{2} " {3} " " " K: $(\sigma_1\varsigma_1[\sigma_2\varsigma_3 - \omega^2\sigma_3\varsigma_2],\ \sigma_2\varsigma_2[\sigma_3\varsigma_1 - \omega^2\sigma_1\varsigma_3],\ \sigma_3\varsigma_3[\sigma_1\varsigma_2 - \omega\sigma_2\varsigma_1])$.

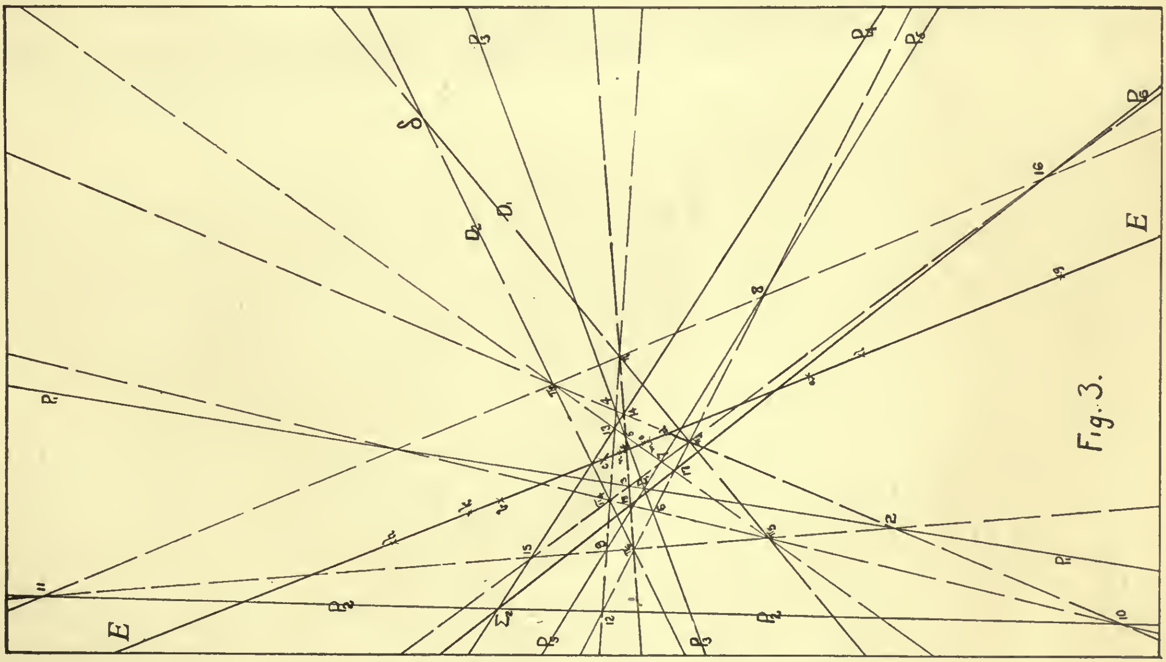
The line $\overline{IJ}$ is the real line D_1 given by (35).

The line $\overline{HK}$ is the real line D_2 given by (37).

Therefore, D_1 and D_2 are the diagonals of the imaginary quadrilateral of the Hessian pairs of the line-triads on Σ_1 and Σ_2.



Hexagon of three equispaced lines on each of two points. Theorem VI.



Hexagon of three points on each of two lines. Theorem I.

Theorem III. follows without further proof.

37. As may be seen from Fig. 3, for the reverse or dual process the three line-triads of a set are mutually in triple perspective thus:

Triangles.	*Centers.*	*Axes of Perspective.*
1 6 8 2 4 9	Σ_1, a, b.	P_2, D_2, D_1.
1 6 8 7 3 5	c, Σ_1, d.	D_2, P_6, D_1.
2 4 9 5 7 3	e, f, Σ_1.	D_2, D_1, P_4.
10 14 18 16 11 15	g, Σ_2, h.	D_2, P_1, D_1.
11 15 16 17 12 13	k, Σ_2, l.	D_2, P_5, D_1.
10 14 18 12 13 17	Σ_2, m, n.	P_3, D_2, D_1.

The points a, b, c, ... m, n are the twelve points on E.

38. Without further proof because they follow analytically from data already given and may be tested in Fig. 3, we present several theorems:

IV.

A triangle of one set of three is in two-fold perspective with any one of the opposite set of three triangles, but for all such perspectives—

there are only nine axes each taken twice and situated on the covariant point ε. The centers in each case are Σ_2 and Σ_1.

there are only nine centers each taken twice and situated on the covariant line E. The axes in each case are D_2 and D_1.

V.

The point- and line-triads are between themselves in *single* perspective. The center of perspective in each case is Σ_1 if the two triads are both of the first or both of the second set of three as herein classified, and Σ_2 is center if they are of oppositely named sets.

39. The special forms or arrangements for the three lines on each of two points to which attention is called are,

(1) Two sets of equispaced triads; i. e., lines at angles of $\frac{2\pi}{3}$.

(*a*) The points being the two equiangular points of the triangles of one set of three.

(*b*) The points being on the circumcircle of equiangular triads.

(2) The points taken at infinity,

(*a*) arbitrarily, giving two sets of three parallel lines each, at an angle ϑ with each other.

(*b*) at the circular imaginary points of the plane, giving two sets of perpendicular lines.

These various forms are handled analytically best by using special coördinate systems for the several cases, and furnish nice work in devising expeditious methods. I omit as irrelevant the various methods that were employed.

40. From these special forms we have the theorems,

VI.

For a triad of equispaced lines on each of two points, one set of the three point-triads consists of equiangular triangles with sides respectively parallel (Fig. 4). Thus one of the lines D is at infinity and the other is perpendicular bisector of the line S between Σ_1 and Σ_2. Further, the circumcircles of the three equilateral triads pass through Σ_1 and Σ_2, and those of the other set of three triangles intersect in ε, the pole of S as to any of the six triangles.

VII.

If to the conditions of the previous theorem we add that one of the set of scalene triangles is also equiangular, then the vertices of the other two triangles of its set are inverse points as to its circumcircle, and all the circumcenters of the set of three equilateral triangles are on the finite Hessian diagonal D.

The proof of this last theorem is especially neat by use of circular coördinates

Finis.

VITA.

Charles Clayton Grove was born December 19th, 1875, the son of Lewis and S. Elizabeth Grove, at Hanover, Pa. His early education was received in the private school of Miss Martha E. Grove and in the public schools of Hanover.

In 1896, he was graduated from the Millersville (Pa.) State Normal School, where he prepared for college. After two years of teaching, he entered in September, 1898, upon the classical course of Pennsylvania College, Gettysburg, receiving the degree of Bachelor of Arts in 1900. The next winter he was Supervising Principal of the schools of Hummelstown, Pa. In 1903, the degree of A. M. was conferred at Gettysburg.

In October, 1901, he entered upon graduate study at the Johns Hopkins University with Mathematics, Physics and Italian as subjects. At its very close, this course was interrupted by his taking an instructorship in Mathematics in Pennsylvania State College. During the winter of 1905–6, he was an instructor at the Baltimore Polytechnic Institute, and completed the work for his doctorate at the Johns Hopkins University.